FINALS II 1982-83

1. Jones runs at 4/3 the rate of Smith, and Allen runs at 9/8 the rate of Jones. In a 1000 meter race Jones runs against the relay team of Smith and Allen.

(a) If Smith and Allen each run 500 meters, who is the winner? (Jones or the relay team of Smith and Allen)

(b) If the race is to end in a tie, how much of the 1000 meters must Smith run?

2. (a) Estimate the number π by first computing the area of a 12-sided polygon inscribed in a circle.

(b) In (a) using instead a circumscribed regular 12-sided polygon, show π is estimated by $12(2 - \sqrt{3})$. (Another form of the answer is $12[7 - 4\sqrt{3}]^{1/2}$. This value is approximately 3.3288.)

3. (a) Determine the number of ways the number 216 can be written as a product of three positive integers. (Different orderings of the same factors, such as 1x2x108 and 2x1x108 are to be considered the same product).

(b) Repeat (a) for the number 1000.

4. Given that the equation $x^3 + x^2 - 2x + D = 0$ has three real roots in geometric progression, find D.

5. In a double elimination tennis tourament (a player is eliminated after two defeats), there are 48 players. How many matches will be played before the tournament is over? Give all possible answers.

6. Show that if $a_1, a_2, ..., a_n$ are distinct positive integers not divisible by any primes other than 2,3,5 then

$$1/a_1 + 1/a_2 + ... + 1/a_n < 15/4.$$

7. Let $a_1, a_2, ..., a_n$ be an arbitrary arrangement of the numbers $1,2,...,n$. Prove that, if n is odd, then the product $(a_1 - 1)(a_2 - 2)(a_3 - 3)...(a_n - n)$ is an even number.

8. Urn *I* and urn *II* each have 1 red and 4 black balls. Urn *III* has R red and B black balls. Three balls are drawn at random from urn *I* and placed in urn *II*. Then three balls are drawn at random from urn *II* and placed in urn *III*. Finally one ball is drawn at random from urn *III*.

(a) Find the relation between R and B in order that the ball drawn from urn *III* is more likely to be red than black.

(b) Repeat (a) if urn *I* and urn *II* each have 2 red and 8 black balls.