FINALS III 1983-84

1. Determine the number of integers from 1 to 999 which contain either the digit 3 or the digit 5, and such that the middle (tens unit) digit is an odd integer.

2. A woman has at least 2 children and more than $300 in the bank. If the product of her age, the number of dollars in her account and the number of children she has (all integers) is 31310, how many children does she have, what is her age, and how much money does she have in the bank?

3. If A,B,C are positive integers then by definition $A = B \bmod C$ provided $\dfrac{A-B}{C}$ is an integer.

(a) Prove if $A_1 \equiv \bmod C$ and $A_2 \equiv B_2 \bmod C$ then $A_1A_2 \equiv B_1B_2 \bmod C$

(b) Prove that if $A \equiv B \bmod C$ then for every positive integer n, $A^n \equiv B^n \bmod C$.

4. Given that the equation $x^5 + 4x^4 + Cx^3 + Dx^2 + Ex + F = 0$ has $x = i$, where $i^2 = -1$, and $x = 3$ as two of its roots and 30 for the product of the roots:

(a) Find the values C,D,E,F.

(b) Find the remaining roots.

5. George starts with the amount 256 cents and makes a series of bets. On each bet he either wins or loses half of his amount; however if this results in a fractional (non-integer) number then he loses all of his amount (his amount is then 0).

(a) Could he ever have the amount 25 cents? If so show a sequence of wins and losses to reach that amount; if not give a proof or explanation.

(b) Repeat (a) if the amount is 27 cents.

(c) Give a general formula for the possible values (other than 0) for George's amount at some stage in the betting.

(d) Give the number of possible amounts in (c).

6. In a race Tom, Bill, and Jack run at constant speeds. Tom beats Bill by 2 miles and Tom beats Jack by 3 miles. If Bill runs 10/9 as fast as Jack, and it takes Bill 1 hour to run the race, how long does it take Tom?

7. (a) Show that if A is an integer then either $A^2 \equiv 0 \bmod 4$ or $A^2 \equiv 1 \bmod 4$ (see problem 3 for definition of $A \equiv B \bmod C$).

(b) Show that there are no positive integers x,y,z for which

$x^2 y^2 - x^2 - y^2 = z^2$. (Hint: Consider what happens mod 4)