FINALS IV 1984-85

1. Joe and Tom run a race. Joe runs at a rate of 1/5 mile per minute during the first minute; his rate is reduced by 1/10 of his current rate at the end of the first minute and at the end of each minute thereafter. Tom runs a constant rate of 1/6 mile per minute.

(a) Find the integer n such that Joe is ahead after n minutes and Tom is ahead after n + 1 minutes.

(b) Let the values 1/5, 1/6,and 1/10 in (a) be respectively replaced by r_j , r_t, and m where $r_j > r_t$. Write an equation in terms of r_j , r_t , m and n which, if solved for n, could be used to compute the value as found in (a). Simplify the equation but <u>do not try to solve for *n*.</u>

2. Let the transformation T on the set of positive integers be defined by

$$T(N) = \frac{N}{3} \quad \text{if } N \text{ is divisible by 3}$$
$$= \frac{N-1}{3} \quad \text{if } N - 1 \text{ is divisible by 3}$$
$$= \frac{N-2}{3} \quad \text{if } N - 2 \text{ is divisible by 3.}$$

(a) Find all values of N such that $T(N) = 6$.

(b) Define $T^2(N) = T(T(N))$, $T^3(N) = T(T^2(N))$,...,$T^K(N) = T(T^{K-1}(N))$ for K = 2,3,... and N = 1,2,... . Let S be the set of all integers N such that $T^5(N) = 6$. Which numbers are in S?

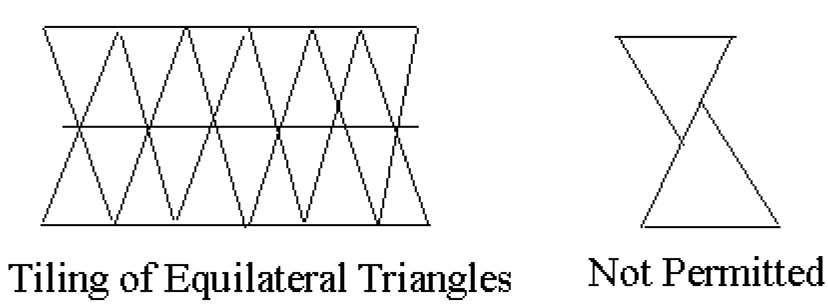
(c) With S as in (b), let R be the set of numbers of the form $T^2(N)$ where N is in S. Which numbers are in R? Give a simpler method for describing the set R (not using the set S).

3. A regular n-polygon is a polygon with n sides of equal length; the interior angles are also equal.

(a) In degrees, what is the measure of each interior angle of a regular n-polygon?

(b) Prove that if a floor is to be covered with a regular n-polygon tiling, then either $n = 3$, $n = 4$, or $n = 6$.

(Note: a vertex of one tile must coincide with a vertex of an adjacent tile i.e. cannot lie interior to an edge)



4. By the greatest common divisor (GCD) of two positive integers is meant the largest integer which divides both integers. Let M,N,A be positive integers and G the GCD of M,N and H the GCD of $M,M + AN$.

(a) Prove $G \leq H$.

(b) Give an example in which $G < H$.

(c) Prove that if the GCD of N,A is 1 then $G = H$.

(d) Give a necessary and sufficient condition for $G = H$. Prove your result. Also give an example to show the condition in (c) is not necessary.

5. An urn contains 2 red and 4 black balls.

(a) A ball is drawn from the urn; if it is black then it is returned to the urn, but if it is red then it is not returned to the urn. A second ball is then drawn from the urn. What is the probability that one of the balls drawn was red and the other black?

(b) A ball is drawn from the urn N times in succession; each time the ball is replaced if it is black but not replaced if it is red. A ball is then drawn from the urn. In terms of N, what is the probability that the ball is red?

(c) In (b) is the probability ever equal to zero? Explain