FINALS V 1985-86

1. Bill and Joe run a race. Bill starts at a rate of .2 miles per minute and at the end of each minute reduces his rate by .01 miles per minute. Joe starts at a rate of .1 mile per minute and at the end of each minute increases his rate by .01 mile per minute.

(a) How far do Bill and Joe go in the first 5 minutes?

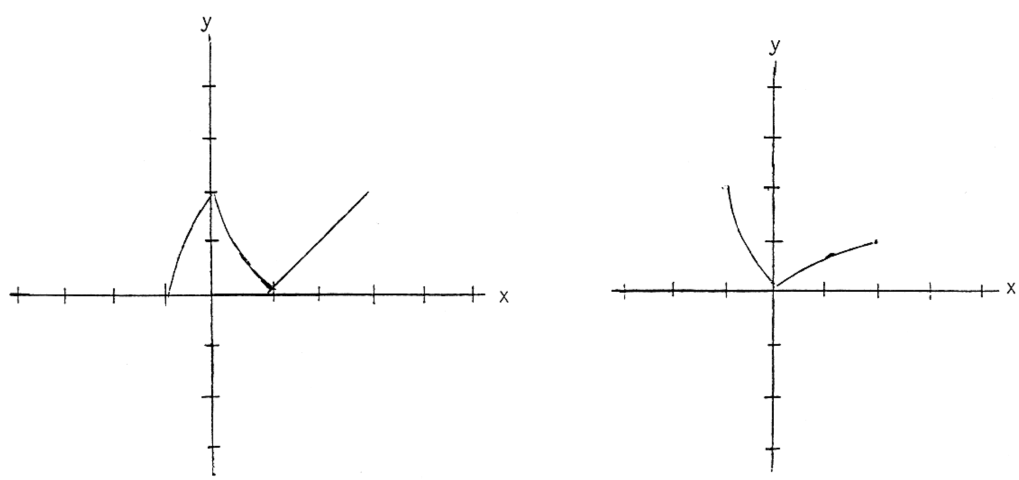
(b) How far would they run before Joe catches Bill?

(c) Suppose Bill starts at a rate of r miles per minute and decreases his rate at the end of each minute by d miles per minute, while Joe starts at a rate of s ($s < r$) miles per minute and each minute increases his rate by e miles per minute. In what minute interval, in terms of r,s,d,e will Joe catch Bill?

2. Given a triangle whose exterior angles are in the ratio 5:9:10 and whose shortest side has length 20, find the area of the triangle.

3. (a) In the figure is the graph of $y = f(x)$; on the same set of axes sketch the graph of $y = f(2 - x) + 1$.

(b) In the figure is the graph of $y = f(x)$. If $f^{-1}(x)$ is the inverse function of $f(x)$, on the same set of axes sketch the graph of $f^{-1}(x/2)$.



(a) (b)

4. The number 1 is initially on blackboard A and the number 0 on blackboard **B**. The number on B is replaced by the difference of the numbers on B and A (i.e. 0 - 1 = -1 is the first replacement). The number on A is then replaced by the previous number on B (i.e. 0 is the first replacement). This process is performed N successive times.

(a) Find the numbers on blackboards A and B for N = 1,2,3, 4,5,6,7,8,9. (For $N = 1$ the values are 0 and -1).

(b) Find a formula for the number on A and B for an arbitrary positive integer N.

(c) Find the numbers on A and B for $N = 1{,}000$.

5. Urn A has two red balls and one black ball; urn B has two black balls and one red ball. Initially a ball is drawn from urn A; if the ball is red, it is placed in urn B and the next ball is drawn from urn B. If it is black it is returned to urn A and the next ball is drawn from urn A.

(a) What is the probability the second ball drawn is red?

The process is successively repeated as follows: if a red ball is drawn from urn A or urn B then it is placed in the other urn and the next ball is drawn from the other urn; if a black ball is drawn then it is returned to the urn and the next ball is drawn from the same urn.

(b) What is the probability the third ball drawn is red?

(c) Let $P(N)$ be the probability the Nth ball is drawn from urn A. Show $P(N) = \dfrac{3+P(N-1)}{6}$, $N = 2,3,...$ and solve using the initial condition $P(1) = 1$ to obtain a formula for $P(N)$.

(d) Find the probability the Nth ball drawn is red.

6. In this problem a 'perfect square' will mean the square of a positive integer; neither 0 nor negative integers should be considered in this problem.

(a) Give a positive integer $D > 2$ which cannot be expressed as the difference of two squares.

(b) Give distinct perfect squares M,N,P,Q such that $M - N = P - Q$.

(c) Let $D > 2$ be a positive integer. Give, with proof, a necessary and sufficient condition that D can uniquely be expressed as a difference of two perfect squares.