FINALS VI 1986-87

1. Given that the equation $x^3 + mx^2 + nx + p = 0$ has integer roots, where m,n,p are integers:

(a) Prove that if p is odd then m is odd.

(b) In the table below write YES in each position that the odd/even property of that column implies the odd/even property of that row. Prove each case you have written YES.

	m odd	m even	p odd	p even	n odd	n even
m odd	XXX	XXX	YES			
m even	XXX	XXX				
p odd			XXX	XXX		
p even			XXX	XXX		
n odd					XXX	XXX
n even					XXX	XXX

(c) Give examples of m,n,p which show that if m is even and p is even then n may be either odd or even.

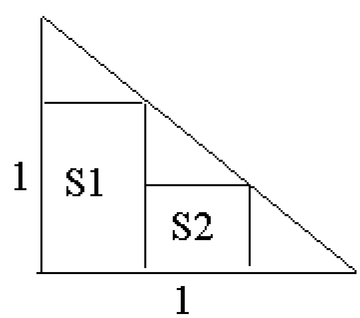
(d) As in (c), give examples of every case in which the odd/even properties of two of m,n,p does not necessarily imply the other is even or odd.

2. (a) Let $S(x,y) = y + xy - x^2 - y^2$. Show S may be written in the form, for suitable numbers a,b,c,d:

$$S(x,y) = a - b(my - x)^2 - c(y - d)^2 .$$

(b) In (a) find the maximum value of $S(x,y)$ for all real numbers x,y.

(c) In the figure, find the maximum value $S_1 + S_2$ of the sum of areas of pictured rectangles. (You may wish to use the results of (a) and (b)).



3. An urn contains a red ball, a green ball, and a blue ball. A ball is drawn at random and returned to the urn. This process is repeated until every ball has been drawn at least once. Let N be the number of draws required, and let $P(N)$ be the probability that N draws were required to draw each ball at least once.

(a) Find the values $P(1)$, $P(2)$, $P(3)$, $P(4)$.

(b) Find a formula for the value $P(N)$, N arbitrary.

(c) For N arbitrary find the probability $L(N)$ that at least N draws are required until each ball has been selected. (Write answer in closed form).

4. Bill plays roulette and bets \$5 on his first bet; thereafter

(i) If he won the previous bet then he bets \$5 on the next bet.

(ii) If he lost \$N on the previous bet then he bets \$(2N - 4) on the next bet.

Assuming the payoff is the next bet:

(a) If Bill bets 6 times and wins 2 of the times, what are the possible net amounts he could win or lose?

(b) If Bill wins on his last bet, does his net winnings or losses depend on the order of his previous wins and losses or only on how many previous wins and losses?

(c) Repeat (b) if Bill loses on his last bet.

(d) If Bill wins 20 times and loses 24 times, and won on his last bet, what are the possible net amounts he could win or lose?

(e) If Bill wins 23 times, loses 19 times, and loses \$8 on the last bet, what are the possible net amounts he could win or lose?

5.(a) Show that $1/11! + 1/12! + 1/13! + ... + 1/30! < 1/10!$.

(b) Let $S = r^N + r^{N+1} + r^{N+2} + ... + r^{N+30}$ where $0 < r < 1$. Find a value of N, expressed in terms of r, such that $S < .0001$.

(c) Let e be a small positive number. Find a value of N in terms of e such that

$$1/N! + 1/(N + 1)! + 1/(N + 2)! + ... + 1/(N + K)! < e$$

for all positive integers K.

6. For $T = \{1,2,3,...,N\}$ and $1 \le K \le N$ let $S(N,K)$ be the sum of all alternating sums of the K numbers

$$a_1 - a_2 + a_3 - a_4 + ... + (-1)^{K+1} a_K \quad \text{where } a_1, a_2, ..., a_K \text{ are in } T \text{ and } a_1 > a_2 > ... > a_K.$$

For example

$$S(4,2) = (4 - 3) + (4 - 2) + (4 - 1) + (3 - 2) + (3 - 1) + (2 - 1) = 10$$

$$S(4,3) = (4 - 3 + 1) + (4 - 3 + 2) + (4 - 2 + 1) + (3 - 2 + 1) = 10.$$

(a) Find $S(5,3)$.

(b) Give the number of alternating sums in $S(N,K)$.

(c1) For $K < N$ write a reduction formula which results from the separation of $S(N,K)$ into those alternating sums whose first term is $< N$. (i.e. find a formula for evaluating $S(N,K)$ if values of $S(N',K')$ were known for all $N' < N$ and $K' \le K$.

(c2) For $K < N$ write a reduction formula which results from decreasing each number in the expression for $S(N,K)$ by 1. This produces a separation of $S(N,K)$ into those alternating sums whose smallest term is 1 and those whose smallest term is < 1.

(d) From (c1) and (c2) find a formula for $S(N,K)$.