

FINALS VII 1987-88

1. (a) Given that $0 < a, b, c, d < 1$ prove
$$(1 - a)(1 - b)(1 - c)(1 - d) > 1 - a - b - c - d$$

(b) It is known that of all rectangular parallelopipeds inscribed in a sphere the one having the maximum volume is a cube. Using this fact describe the rectangular parallelopiped, inscribed in a hemisphere, which has maximum volume. Explain your answer.

2. A monkey tries to climb a greasy 100 foot high pole. Starting at the bottom it climbs 1 foot, then slides down $\frac{2}{3}$ of the distance it has just climbed, and then climbs $\frac{5}{3}$ of the distance it has just slid down. The monkey thereafter repeats the process until, if ever, it reaches the top of the pole (i.e. it alternately slides down $\frac{2}{3}$ of the distance it has just climbed and climbs $\frac{5}{3}$ of the distance it has just slid down).
 - (a) How high is the monkey after the first 2 climbs (do not count the initial 1 foot climb)?
 - (b) In terms of n , how high is the monkey after the first n climbs (do not count the initial 1 foot climb)?
 - (c) Will the monkey ever reach the top of the pole? If so justify your answer and give an integer estimate of how many climbs are required. If not, describe the approximate position of the monkey after a very large number of climbs and downslides.

3. (a) An urn has 4 balls numbered 1,2,3,4. Two balls are drawn (without replacement) from the urn and S is the sum of the numbers on the two balls. Find the probability S is an even number.
 - (b) Repeat (a) if the urn has 5 balls numbered 1,2,3,4,5.
 - (c) Repeat (a) if the urn has N balls ($N > 3$) numbered 1,2,..., N .
 - (d) Repeat (c) if three balls are drawn (without replacement). Assume $N > 5$.
 - (e) Suppose the urn has 100 balls numbered 1,2,...,100 and 13 balls are drawn without replacement. Prove the probability is $\frac{1}{2}$ that the sum on the balls is even.

4. (a) A right triangle has length $S = 1$ for one of the legs and length x for the other leg. Is there a value of x such that the perimeter of the triangle equals the area?
 - (b) Repeat (a) if $S = 10$.
 - (c) In (a) find all values of S , if any, such that the perimeter of the triangle equals the area.
 - (d) Find all right triangles, each of whose sides has integer length, such that the perimeter equals the area.

5. (a) A finite sequence of integers has the property that the sum of any 3 successive integers is positive and the sum of any 5 successive integers is negative.
 - (a1) Explain why that starting with the fourth number in the sequence the sum of any 2 successive integers must be negative.
 - (a2) Explain why that starting with the sixth number in the sequence every integer must be positive.
 - (a3) Explain why the sequence cannot have more than 6 integers.

(a4) Show by example that the sequence can have 6 integers.

(b) Suppose a finite sequence of integers has the property that the sum of every 7 successive integers is positive and the sum of every 11 consecutive integers is negative. Find a number n such that the sequence can have at most n integers. (the smaller the value of n the more points your solution will receive; do not attempt to find an example for your n)