FINALS VIII 1988-89

1. A father is making financial plans to send his small son to college. Each January 1 he will deposit a sum of money into an account which pays 10% interest compounded annually each December 31. The amount deposited will be increased 10% each year. If $P is the amount deposited the first year then

(a) Give the value, in terms of P, of the account at the end of the second year.

(b) Give the value, in terms of P, of the account at the end of N years.

(c) If the value of the account is to be $150,000 at the end of 15 years, give the value (to the nearest $100) of the first and last deposits.

2. Let $x = 10^{-20}$, $y = 10^{-30}$, $z = 10^{-40}$:

(a) Which is closer to 1: $(1 + x)(1 + y)(1 - z)$ or $(1 + x)(1 - y)(1 + z)$?

(b) Which is closer to 1: $(1 + x)(1 - y)(1 - z)$ or $(1 - x)(1 + y)(1 + z)$?

3. Among all quadrilaterals inscribed in a unit square, determine all of those, if any, whose area is exactly 1/2. (Note: There is a vertex of the quadrilateral on each of the 4 sides of the square; these cannot be vertices of the square).

4. The integers from 1 to 1,000 are written in order around a circle.

(a) Starting at 1 every 14th number is marked (that is 1,15, 29,etc). This process is continued until a number is reached which has already been marked. How many different numbers are marked?

(b) In part (a) let 14 be replaced by N, where $1 < N < 1{,}000$. Obtain a formula, in terms of N, which determines how many different numbers are marked. Hint: Your formula may involve prime factors, least common multiples, greatest common divisors, modulo relations, etc.

(c) Using the formula in (b) how many different numbers are marked for $N = 15$? for $N = 16$? for $N = 17$? for $N = 375$?

5. An urn has 2 red balls and 1 black ball. A ball is drawn from the urn, and then returned to the urn if and only if it is black; this is then repeated for an indefinite number of times. Let $P(N)$ be the probability the Nth ball drawn is red.

(a) Find $P(2)$, $P(3)$, $P(4)$.

(b) Find a formula for $P(N)$; express the answer in closed form (without extended sums).

(c) Find the smallest integer N such that $P(N) < 0.001$.