FINALS IX 1989-90

1.(a) The graph of the line $y = 7x + 3$ is shifted 5 units to the right, then shifted 4 units down, and then rotated 45° clockwise about the origin. Find the equation of the resulting graph.

(b) The graph of the equation $f(x,y) = 0$ is reflected about the x axis, and then rotated 90° counterclockwise about the point (h,k). Find the equation of the resulting graph (answer should be expressed in terms of f,x,y,h,k).

2. Let n,k be positive integers.

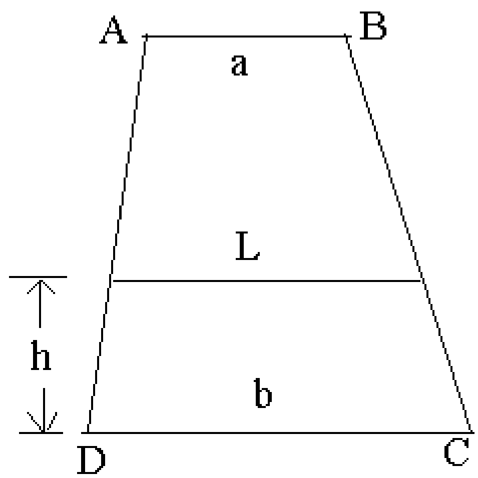
(a) Prove that at least one of the integers n, $n + 4$, $n + 8$ is divisible by 3.

(b) Prove or disprove that at least one of the integers n, $n + 2^k$, $n + 2^{k+1}$ is divisible by 3.

3. Given a trapezoid $ABCD$ which has parallel sides AB and DC and area 1, let a be the length of AB, b the length of DC, and let L be the length of the segment EF parallel to AB and DC, where E is on AD, F is on BC, and the distance from EF to DC is h.

(a) Find a formula for L in terms of a,b,h.

(b) Find a necessary and sufficient condition, in terms of a and b, so that $L \geq h$ in all cases.



4.(a) From a list $S_3 = \{a,b,c\}$ of three (not necessarily distinct numbers) one can generate a list $T_3 = \{a + b, a + c, b + c\}$ of three numbers by adding pairs of numbers in S_3.

(a1) Given that $T_3 = \{17,29,44\}$ find a list S_3 that generates T_3.

(a2) Prove that for every list T_3 there is a unique list S_3 that generates T_3.

(b) From a list $S_4 = \{a,b,c,d\}$ of four (not necessarily distinct numbers) one can generate a list $T_6 = \{a + b, a + c, a + d, b + c, b + d, c + d\}$ of six numbers by adding pairs of numbers in $S_4 = \{18,21,26,29,34,37\}$

(b2) Prove that not all lists T_6 of six numbers can be generated by a list S_4. If $T_6 = \{u,v,w,x,y,z\}$ find a necessary and sufficient condition in terms of u,v,w, x,y,z such that T_6 can be generated by a list S_4

5.(a) Find integers x,y such that $2x + 3y$ is an integer multiple of 13; show that for these values of x,y it is true that $7x + 4y$ is also an integer multiple of 13.

(b) Prove that if x,y are any integers such that $2x + 3y$ is an integer multiple of 13 then $7x + 4y$ is also an integer multiple of 13.

(c) Find integers m and n, where $3n \neq 5m$, such that if $3x + 5y$ is an integer multiple of 23 then $mx + ny$ is also an integer multiple of 23.

6. An urn has N ($N > 2$) balls numbered 1,2,3,...,N. Three balls are drawn from the urn (no balls are put back in the urn). Let $P(N)$ be the probability that one of the three numbers drawn is greater than or equal the sum of the other two.

(a) Find $P(3)$, $P(4)$, $P(5)$, $P(6)$.

(b) Find an expression for $P(N)$. For maximum credit write the expression in closed form (no extended sum or difference expressions). You may wish to use the sum formulas:

$$1^2 + 2^2 + ... + k^2 = k(k + 1)(k + 2)/6$$

$$1x2 + 2x3 + 3x4 + ... + kx(k + 1) = k(k + 1)(k + 2)/3$$