FINALS X 1990-91

1. A man has initially \$*S* in his piggy bank. The first month he takes out half of his money, and the second month he adds \$*D*. The third month he takes out half his money, and the fourth month he adds \$*D*. This continues, every other month taking out half his money, and every other month adding \$*D*. In terms of *S* and *D* how much money is in the bank at the end of
(a) 1 year? (b) 10 years? (c) *n* years? Also (d) is the amount in the piggy bank getting very large, very small, or varying in some other way in terms of *S* and *D* after a very long period of time? Be as precise as possible in your answer.

2. The Lakers and Celtics are to play for the basketball championship. The first team to win four games is the winner. The Lakers are the home team for games 1,2,6,7 and the Celtics for games 3,4,5 (if one team has won four games the remaining games are not played). If the Lakers are the home team their probability of winning is r and if the Celtics are the home team the Lakers probability of winning is s $(0 < r,s < 1)$.

In terms of r and s find the probability the Lakers win in exactly (a) four (b) five (c) six games. Also (d) if $r = 2/3$ and $s = 1/3$ find the probability the Lakers win in exactly seven games. In addition (e) if the Lakers were the home team for some other set of four games (e.g. games 1,2,3,4 or games 1,3,5,7) would the probability the Lakers win the series be the same? Explain your answer.

3. (a) Given the quadrilateral *ABCD* with *AB* parallel to *CD* and P the midpoint of *DC*, suppose $AB = a$, $AD = BC = b$ and x is the distance from P to each of A,B,C,D. Find a formula for x in terms of a and b.
(b) A hexagon is inscribed in a circle. If two opposite sides have length 2 and the other four sides have length 1 find the area of the circle.

4. (a) Give a sequence of integers a,b,c,d such that none of the following sums is divisible by 4: $a + b$; $b + c$; $c + d$; $a + b + c$; $b + c + d$; $a + b + c + d$.
(b) Let $a_1, a_2, a_3, \ldots, a_n$ be a sequence of n integers. Let an interval sum be defined to be any sum of the form $a_j + a_{j+1} + a_{j+2} + \ldots + a_k$ where $1 \le j \le k \le n$. For each integer m, $1 \le m \le n$ prove there is an interval sum which is divisible by m.

5. Prove the following inequalities for any positive numbers a,b,c,d. You are permitted to use (a) to prove (b), and to use (b) to prove (c).

(a) $a + b \ge 2(ab)^{1/2}$
(b) $a + b + c + d \ge 4(abcd)^{1/4}$
(c) $a + b + c \ge 3(abc)^{1/3}$