FINALS XIII 1993-94

1. Using the two operations (i) double and (ii) add one

(a) Find the least number of operations to convert 1 to 63 and show the corresponding sequence. (for example 1,2,4,5 uses 3 operations to convert 1 to 5).

(b) For a given positive integer K give an algorithm (step by step procedure) for converting 1 to K using the least number of operations and prove it gives the least.

(c) Using the algorithm in (b) find the least number of steps to convert 1 to 100.

(d) For N a positive integer find the least number of operations to convert 1 to 2^N -1.

2. Given trapezoid $ABCD$ with AB and CD parallel, and the diagonals AC and BD intersect in point E. If the area of triangle AEB is 20 and the area of triangle CED is 10, find the area of the trapezoid.

3. Eight different colored balls are placed in boxes numbered 1 through 8, one ball to each box. The balls are then transferred to different boxes as follows:

Original Box	1	2	3	4	5	6	7	8
New Box	4	3	8	7	6	5	1	2

The balls are then again transferred exactly as before, that is the ball in box 1 goes to box 4, in box 2 goes to box 3, etc. The transfers are repeated until each ball is back in its original box.

(a) What is the least number of transfers required to return each each ball to its original box?

(b) Show that for any different transfer, if that same transfer is repeated sufficently many times then eventually all balls will be back in the original box.

(c) What transfer would need a maximum number of repetitions to return all balls to their original box? In this case how many repetitions are required?

(d) Find a number N such that for N repetitions of any transfer necessarily all balls will be back in their original box.

(e) In (d) find the smallest possible value of N and prove it is the smallest.

4. Let m and n be distinct positive integers.

(a) Find three pairs of values for m and n such that $2^m + 2^n$ is a perfect square.

(b) Show there are infinitely many pairs of values for m,n such that $2^m + 2^n$ is a perfect square.

(c) Show $4^m + 4^n$ cannot be a perfect square.

(d) Show $5^m + 5^n$ cannot be a perfect square.

5. An object starts at the point (0,0) and each unit of time moves either one unit to the right or one unit up. The probability it moves to the right is r. If it has moved m units to the right and n units up after $(m + n)$ moves then its coordinates are (m,n) at that time. Find the probability in terms of r that the object passes through the point

(a) (1,2)

(b) (4,2)

(c) (m,n) where m and n are arbitrary non-negative integers.

Also

(d) Find the probability the object passes through both of the points (1,2) and (5,4).

(e) Find the probability the object passes through (5,4) but does not pass through (1,2).

(f) Find the probability the object moves 4 units to the right before it moves 3 units up.

(g) Team *A* plays Team *B* a series of games; each game the probability Team *A* wins is r. What problem related to wins and losses is equivalent to problem (f)?