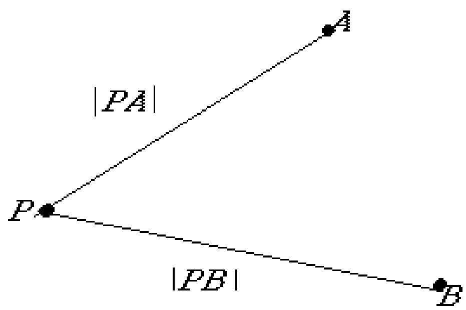
FINALS XIV 1994-95

1. Find all sets of 5 positive integers whose sum equals their product; prove there are no others. Note: The 5 integers are not necessarily different.

2. Let A and B be two distinct points in the plane. Describe the geometric curve which is the locus of all points P such that $\sec \angle APB = |PA|\,|PB|$. Note: sec is the secant function from trigonometry.



3. In the diagram below, an object starts at 0 and moves to the right (i.e. always in an east, southeast or northeast direction). There is one path 01 to 1, three paths 012, 02, and 01'2 to 2, and four paths 013, 0123, 023, 01'23 to 3.

(a) Find the number of paths to 4 and to 5.

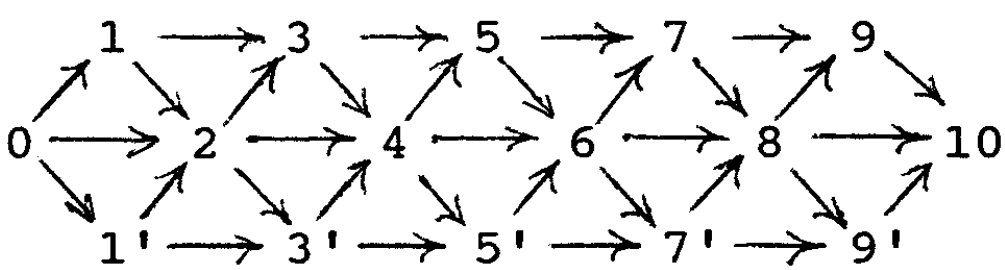
(b) Determine a method using formulas or an algorithm for computing the number of paths $P(N)$ from 0 to N for each positive integer N.

(c) Using your solution in (b) compute the number of paths from 0 to 8.

Warning: Don't try to list them as there are too many!

(d) Find the number of paths from 0 to 8 which pass through 4.

(e) Find the number of paths from 1 to 8



4. (BOILED EGG PROBLEM) Given an x minute and a y minute sandtimer boil an egg for exactly t minutes (where x,y,t are positive integers).

Note: In some cases there may be a waiting time before starting the boiling of the egg. Also for a given case a solution, if it exists, may not be unique.

In cases (a),(b),(c) find a solution for the Boiled Egg Problem, if it exists.

(a) $x = 3$, $y = 2$, $t = 17$
(b) $x = 5$, $y = 9$, $t = 3$
(c) $x = 3$, $y = 6$, $t = 14$

(d) Find a relation among x,y,t that is necessary and sufficient that there be a solution to the Boiled Egg Problem.

(e) For the cases when there is a solution to the Boiled Egg Problem give an algorithm for finding a solution. Also, if there is a waiting time, your algorithm should show how to find the solution with the least waiting time.

(f) Apply your algorithm to the case $x = 12$, $y = 5$, $t = 6$.

Note: Students may use a calculator on problem 5.

(PROBLEM 5 IS A CALCULATOR PROBLEM)

5. Bill starts with \$ P and makes a series of n bets, each time betting 1/2 of his amount (P + winnings - losses). He wins k of the n bets. He is said to be a *Winner* if he ends with more than \$$P$. Assuming his amounts are calculated to the nearest cent:

(a) For $n = 4$ if Bill wins the first and last bets and loses the second and third bets, find the final amount in terms of P.

(b) Find in terms of k,n,P Bill's amount after the n bets and show it is independent of the order of wins and losses.

(c) If $n = 1000$ find the smallest value of k for Bill to be a *Winner*.

(d) If $n = 1000$ find the smallest value of k for Bill to finish with at least \$($100P$)

(e) Repeat (c) if each time Bill bets 3/4 (instead of 1/2) of his amount.

(f) Repeat (c) if each time Bill bets 1/4 (instead of 1/2) of his amount.

(g) Suppose each time Bill bets the fraction r, $0 < r < 1$ of his amount. In terms of r determine the values of the ratio k/n needed for Bill to be a *Winner*.

(h) In (g) if r is very close to 1 what can be said about the value of the ratio k/n needed to be a *Winner*? (Explain your answer)

(i) In (g) if r is very close to 0 what can be said about the value of the ratio k/n needed to be a *Winner* ?(Explain your answer).