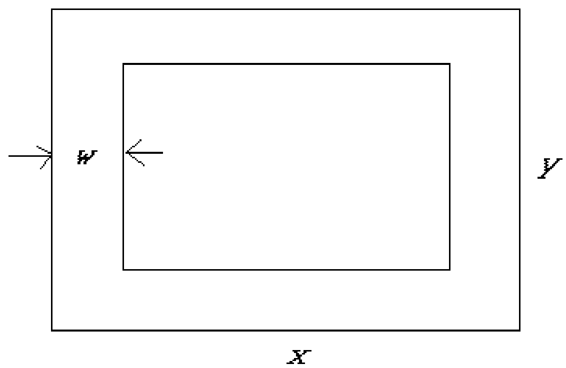
FINALS XV 1995-96

1. In this problem the variables w,x,y denote positive integers. Given the pictured rectangle with sides x and y, and having a border of width w:

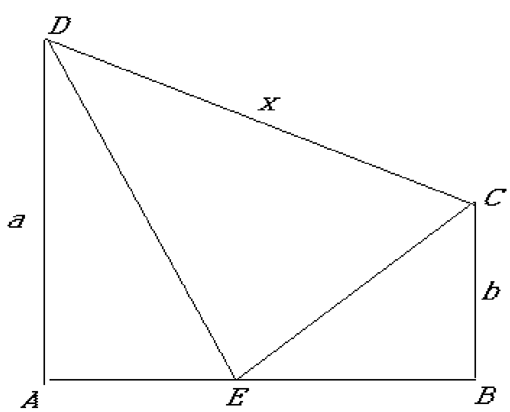
(a) If $w = 1$ find all possible values for x and y, $x < y$.
(b) If $w = 2$ find all possible values for x and y, $x < y$.
(c) For w arbitrary describe in terms of w all possible values for x and y.
(d) Determine all cases, if any, for w,x,y where $x = y$.



2. Let $ABCD$ be a trapezoid with parallel sides AD and BC perpendicular to side AB. Let E be a point on side AB and a,b respectively the lengths of sides AD and BC, where it is assumed $a > b$.

(a) Given that triangle CDE is equilateral find the length x of the sides in terms of a and b.

(b) Find a relationship between a and b in order that there exist a point E on side AB such that triangle CDE is equilateral.



3. In this problem a,b,c,d,k,n denote non-negative integers; $a \equiv b \bmod n$ means $a - b$ is divisible by n.

(a) Prove if $a \equiv b \bmod n$ and $c \equiv d \bmod n$ then $ac \equiv bd \bmod n$.

(b) Find b, $0 \le b \le 421$ such that $3^8 \equiv b \bmod 422$

(c) Find b, $0 \le b \le 421$ so that $3^{16} \equiv b \bmod 161$.

(d) Suppose k is a positive integer such that the exact value of 3^k is out of the range of your calculator. Give an algorithm for finding b, $0 \le b \le k - 1$, so that $3^k \equiv b \bmod k$.

(e) Using your algorithm in (c) find b, $0 \le b \le 421$ such that $3^{422} \equiv b \bmod 422$.

4.(a) An experiment has possible outcomes A or B, where A and B cannot occur simultaneously. If A has probability r and the experiment is repeated until either A or B occurs, find the probability in terms of p and r that A occurs before B.

(b) A bent coin has probability 3/5 heads and 2/5 tails on a given toss. If the coin is repeatedly tossed find the probability that a head followed immediately by a tail (*HT*) occurs before a tail immediately followed by a head (*TH*) occurs.

(c) In (b) find the probability that *TH* occurs before *HH* occurs.

(d) In (b) find the probability that *HH* occurs before *HT* occurs.

(e) Which is most likely to occur first (and why)?

(i) *HT* or *TH* (ii) *TH* or *HH* (iii) HH or *HT*

(f) What is unusual about the conclusions in parts (i), (ii), (iii) of (e)?

5. A monkey climbs a 100 foot greasy pole. It climbs half way to the top and falls 1/4 of the way to the bottom. It then climbs half way to the top from its new position and falls 1/4 of the way to the bottom The climbing continues, each time climbing half of the way to the top from its new position and falling 1/4 of the way to the bottom.

(a) The monkey is 50 feet high after the first climb; find its height after the second and third climb.
(b) Find a formula for its height after the n^{th} climb.
(c) Describe the approximate position of the monkey after the n^{th} climb when n is very large.
(d) Does the monkey ever reach the top of the pole? Explain.