FINALS XVII 1997-98

1. For each positive integer $k > 2$ let S_k be the set of all strings (words) of length k made up from the two letters X and Y such that each three (consecutive) letter substring has at least one X and one Y (thus the words in S_k do not include any subword XXX or YYY). For example $S_3 = \{XXY, XYX, XYY, YXX, YXY, YYX\}$. Let N_k be the number of words in S_3; thus $N_3 = 6$.

(a) Find N_4.

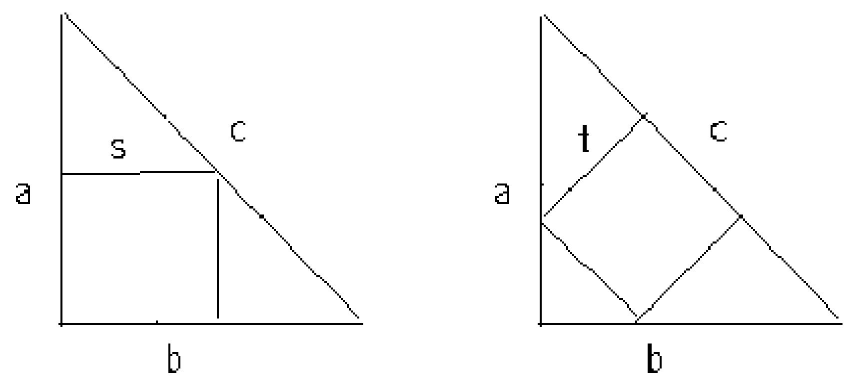
(b) For k arbitrary, $k > 4$, give a method for finding the value N_k from (not necessarily all) the values $N_3, .N_4, ..., N_{k-1}$.
Of course you should give a logical proof for your method.

(c) Using the method from (b) find N_{12}

2. Given the right triangle with sides a,b and hypotenuse c consider the two inscribed squares

(i) square with side s and containing the right angle

(ii) square with side t having one side a segment of the hypotenuse



(a) Find s in terms of a,b and c.

(b) Find t in terms of a,b and c.

(c) Prove that $s > t$ for all values of a,b,c.

<u>3.</u> In the game of baseball a player is said to 'hit for the cycle' if he hits a single, double, triple and home run in the same game. Suppose for a particular player each time he bats the probability he hits a single is s, a double is d, a triple is t, and a home run is h. Also let $n = 1 - (s+d+t+h)$ be the probability of not getting a hit.

(a) If the player bats 4 times in a game what is the probability, in terms of s,d,t,h, he hits for the cycle?

(b) If the player bats 5 times in a game what is the probability, in terms of s,d,t,h,n, he hits for the cycle? Simplify your answer.

In parts (c)-(f) express answer to 3 or more significant digits.

(c) if $s = .18$, $d = .07$, $t = .01$, and $h = .04$ find the probability a player hits for the cycle in a single game if he bats 4 times.

(d) In (c) find the probability the player hits for the cycle at least once if he plays 2,000 games and bats 4 times in each game.

(e) In (c) find the probability the player hits for the cycle exactly 2 times if he plays 3,000 games and bats 4 times each game.

(f) Find the product $p = sdth$ in order that a player have a probability of 1/2 of hitting for the cycle at least once in 2,000 games if he bats 4 times each game.

<u>4</u>. (a) Show that the set {1,2,3,4,5,6} can be divided into three subsets, each of size 2, such that the sums of the numbers in each subset is the same.

(b) Show that the set {1,2,3,4,5,6} can be divided into three subsets, each of size 2, such that the sums of the numbers in the three subsets form an arithmetic progression (of three numbers) with difference 1.

(c) Let K be an arbitrary positive integer > 1. Show that the set $\{1,2,3,...,3K\}$ can be divided into three subsets of size K so that the sums of the numbers of each set is the same.

(d) Let K be an arbitrary positive integer > 1. Show that the set $\{1,2,3,...,3K\}$ can be divided into three subsets of size K so that the sums of the numbers of the three sets form an arithmetic progression (of three numbers) with difference 1.

Hint: Parts (c) and (d) can be done simultaneously)

5. In this problem we consider solutions x,y of the equation

$$** \quad ax = by + c$$ where a,b,c,x,y are all integers.

(a) Find three solution pairs for x,y if $a = 9$, $b = 4$, $c = 3$.

(b) Prove if $x = x_0$, $y = y_0$ are solutions of of ****** and n is an integer then $x = x_0 + nb$, $y = y_0 + na$ are also solutions of ****.**

(c) Prove if a and b are relatively prime and if $x = x_0$, $y = y_0$ are solutions of ****** and $x = x_0 + d$, $y = y_0 + e$ are also solutions of ****** then b divides d and a divides e.

(d) Prove if $a = 1$ or $b = 1$ then ****** has a solution.

(e) Describe infinitely many solutions of the equation $7x = y + 3$.

(f) Determine an algorithm (step by step procedure) for finding a solution to ****** if a and b are relatively prime. It is sufficient to illustrate your algorithm by solving the two equations:

(*f*1) $19x = 9y + 7$

(*f*2) $32x = 27y + 9$

(No credit for an answer for x,y without a description of a general procedure for arriving at the answer.)