DAVID ESSNER FINALS XXI 2001-2002

Use of Calculator: Problems (2(e)) and (3) only.

1. The Sum of Three Reciprocals Problem

Given the equation $1/x + 1/y + 1/z = 1$:

(a) If x,y,z are positive integers, find all solutions of the equation.

(b) If x,y,z are non-zero integers find all solutions in addition to those found in (a).

(<u>In both (a) and (b) you must also prove there are no others</u>).

2. The World Series Problem

Teams A and B play a series of games; in each game either A or B is a winner (there are no ties). The probability that A wins any game is the same regardless of the outcome of the previous games.

(a) If in each game the probability that A wins is 3/5, find the probability that A wins 2 games before B wins 2 games.

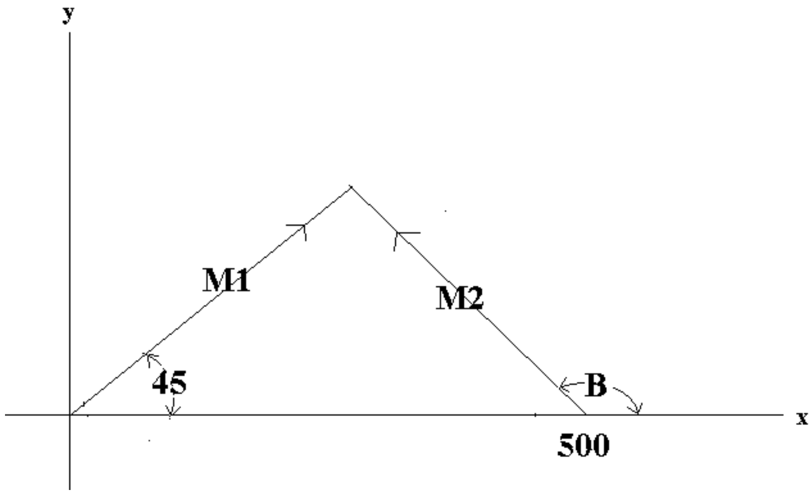
(b) If in each game the probability that A wins is p, $0 < p < 1$, find in terms of p the probability that A wins 2 games before B wins 2 games.

(c) In (b) find in terms of p the probability that A wins 3 games before B wins 3 games.

(d) For what values of p will the probability of A winning 3 games before B wins 3 games be greater than the probability of A winning 2 games before B wins 2 games? You must prove your answer.

(e) (<u>***Calculator Problem***</u>) In the baseball world series the first team to win 4 games is the winner. If in each game team A has probability 3/5 of winning the game, find to two decimal places the probability that team A will win the series.

3. The Missile Interceptor Problem (A Calculator Problem)



A missile **M1** is fired at the origin and travels at a speed of 1000 miles per hour along a straight line (see figure). A second missile **M2** is fired at a speed of ***v*** miles per hour, and at a position on the positive *x* axis and a distance of 500 miles from the origin as shown.

(a) If **B** = 150° and ***v*** = 1500 find, in miles to 4 decimal places, the positions (i.e. (*x*,*y*) coordinates) of **M1** and **M2** one <u>minute</u> after being fired.

(b) If **B =** 150° and ***v*** = 1500 determine to three decimal places at what time in <u>minutes</u> after **M1** is fired that **M2** must be fired in order to intercept **M1.**

(c) If **B =** 150° and **M2** is fired after **M1,** determine to the nearest integer those speeds ***v*** for which **M2** can be fired at some time to intercept **M1.**

(d) If ***v*** = 1500 determine for which angles **B** > 90° it is possible for **M2** to intercept **M1;** express answer to 0.1° accuracy.

(e) If ***v*** = 1500 and **M1**, **M2** are fired at the same time, find **B** in degrees to 3 decimal places if **M2** is to intercept **M1**. Determine in <u>minutes</u> to 2 decimal places the time after **M2** is fired that it intercepts **M1.**

4. The Divisiblity Problem

(a) Prove that $3n^2 + 3n + 1$ is not divisible by either 3 or 5 for any positive integer n.

(b) Prove that none of the numbers 2,3,4,5,6 can divide both $n^3 + 1$ and $(n + 1)^3 + 1$ for any positive integer n.

(c) Find three positive integer values of n such that 7 divides both $n^3 + 1$ and $(n + 1)^3 + 1$ and explain how you arrived at your values of n.

(d) Find all positive integer values of n such that 7 divides both $n^3 + 1$ and $(n + 1)^3 + 1$.

BONUS PROBLEM

(a) Show 7 is the only integer greater than 1 which divides both $n + 2$ and $n^2 - n + 1$ for some positive integer n.

(b) Show 7 is the only integer greater than 1 which divides both $n^3 + 1$ and $(n + 1)^3 + 1$ for some positive integer n.

5. The Tiling Problem

(a) Prove that for all positive integers n, $4^n - 1$ is divisible by 3.

(b) A table is placed on a 1x1 tile in the lower left corner of a 4x4 room (*Figure 2*). Show that the remainder of the room can be tiled with L shaped tiles (*Figure 1*).

(c) If a table is placed on any 1x1 tile in a 4x4 room, show that the remainder of the room can be tiled with L shaped tiles.

(d) Repeat (c) if the table is placed on any 1x1 tile in an 8x8 room.

(e) For every positive integer n, if a table is placed on any 1x1 tile in a room of size 2^n x 2^n then the remainder of the room can be tiled with L shaped tiles; outline the important ideas in a proof of this.

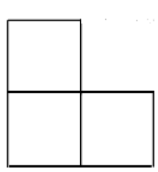


Figure 1

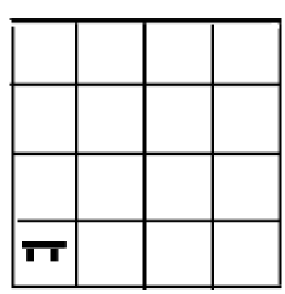


Figure 2