DAVID ESSNER FINALS XXII 2002-2003

The use of a calculator is permitted only on problems 1(c) and 2. Graphic features of a calculator are not permitted.

I Integers Which Are Both Squares and Cubes Problem

For the purpose of this problem let a positive integer be called dual if it greater than 1 and is the square of a positive integer and also the cube of a positive integer.

(a) Find two dual positive integers.
(b) Find with proof a (necessary and sufficient) condition for an integer to be a dual integer.
(c) (Calculator Problem) Find all dual integers less than 600,000.
(d) Do you think the sum of two dual integers could be a dual integer? Why?

II The Modulo Problem (Calculators not permitted)

In this problem if \( n \) is a positive integer then \( n \mod 9 \) denotes the remainder of the division of \( n \) by 9.

(a) Let \( m,n \) be positive integers; prove that if \( n \mod 9 = m \mod 9 \) then \( (n + 1) \mod 9 = (m + 1) \mod 9 \).

(b) Let \( m,n \) be positive integers. Prove that if \( 2^n \mod 9 = 2^{n+m} \mod 9 \) then \( 2^{n+1} \mod 9 = 2^{n+m+1} \mod 9 \).

(c) What are the possible remainders if \( n \) is a positive integer and \( 2^n \) is divided by 9? Justify your answer.

(d) Using (b) find the value of \( 2^{420,000,003} \mod 9 \); explain and justify your method of solution. (Calculators NOT permitted)
III The Ice Cream Cone Problem (A calculator problem)

In the figure $ABC$ is an equilateral triangle with sides of length 2 and $AB$ is a diameter of the pictured semicircle. If the arcs $AD$, $DE$ and $EB$ are equal in length, find

(a) the measure of angle $\angle DCE$ to the nearest 0.1 degree.
(b) the length of $CE$ to the nearest .001

IV The Basketball Ordered Point Sequence Problem

(a) In 1950 a basketball player could score in two ways –
   (i) 2 points for a field goal   (ii) 1 point for a free throw

For $n = 1, 2, 3, \ldots$ let $S_n$ denote the number of ordered ways a player could score $n$ points. For example $S_4 = 5$ since there are the orderings: $(1,1,1,1), (1,1,2), (1,2,1), (2,1,1), (2,2)$.

(a1) Find $S_3$ and $S_5$.
(a2) For $n > 5$ find a method for determining $S_n$ from (not necessarily all of) $S_1$, $S_2$, … , $S_{n-1}$. Explain why your method works.
(a3) Find $S_{11}$ using your method in (a2). Do not try to list the sequences as there are more than 100.

(b) In the present day a basketball player can score points in three ways:
   (i) 3 points for a long field goal   (ii) 2 points for a regular field goal
   (iii) 1 point for a free throw.

Let $T_n$ be the number of ordered ways a player could score $n$ points. For example $T_3 = 4$ since there are the four orderings $(1,1,1), (2,1), (1,2), (3)$. Find $T_{11}$ and explain your method of solution.
V. The Fair Game Gambling Problem

(a) Players A and B alternately roll a 6 sided die, A going first. The first player to roll a 6 is the winner.
   (a1) Find the probability that A wins after A makes 2 or fewer rolls
   (a2) Find the probability A is the winner

(b) Suppose in (a) that A goes first and gets one roll, then B gets two rolls, then A gets one roll, then B gets two rolls continuing so that each time A gets one roll followed by B getting two rolls. What is the probability that A wins?

A gambling game among two or more players is a fair game if each player has equal probability of winning the game.

(c) Suppose A and B play with a die which is ‘loaded’ such that the probability of rolling a 6 is p. If A goes first and gets one try, then B gets two tries, then A gets one try, then B gets two tries, continuing until there is a winner; find the value of p so that this is a fair game.

(d) Suppose A and B play with a die which is loaded such that the probability of rolling a 6 is α and the probability of rolling a 5 is β, where 0 < α < β < 1. A and B alternately roll the loaded die, A going first, and each having one roll at their turn. If A rolls a 6 before B rolls a 5 then A is the winner; otherwise B is the winner.
   (d1) Find the relationship between α and β in order that this is a fair game.
   (d2) Determine the values of α such that there is a value of β to make this a fair game.