DAVID ESSNER FINALS XXIII 2003-2004

The use of a calculator is permitted only on problems 1,2, and 3. Graphic features of a calculator are <u>not</u> permitted . In order to receive credit for numbers obtained by a calculator it is <u>necessary</u> that <u>numerical expressions used to determine these values be displayed.</u>

1. The Compound Interest Problem – A Comparison of Annual and Daily Compoundings

Persons *A* and *D* each invest $1,000 at the same time and at an annual rate of .06 (except in part (b) where the rate is to be determined). *A*'s investment is compounded annually (one time per year) and *D*'s investment is compounded daily (365 times per year).

(a) Find the values of the investments of *A* and *D* after 10 years.

(b) Find to one decimal place the number *n* of years after which the value of *D*'s investment is 5% more than the value of *A*'s investment.

(c) Find to one decimal place the number *n* of years after which the value of *A*'s investment is equal to the value of *D*'s investment after 20 years.

(d) Find the smallest integer *n* such that after n years the value of *D*'s investment is at least $500 more than the value of *A*'s investment.

2. Inscribed Hexagon and Square Problems

(a) As shown in Figure (A), a hexagon, all sides of length **L**, is inscribed in a square whose sides have length 1.

   (a1) Find the <u>exact</u> <u>value</u> and the approximate value to <u>three</u> <u>decimal</u> <u>places</u> of the length **L** of the sides of the hexagon.

   (a2) Find to the nearest <u>0.1</u> <u>degree</u> the measures of the six interior angles of the hexagon.

(b) A regular hexagon (i.e. all sides have the same length and all interior angles have the same measure) is inscribed in a circle of radius 1, and a square is inscribed in the hexagon as shown in Figure B. Determine the <u>exact</u> <u>value</u> and the approximate value to <u>three</u> <u>decimal</u> <u>places</u> of the length of the sides of the square.
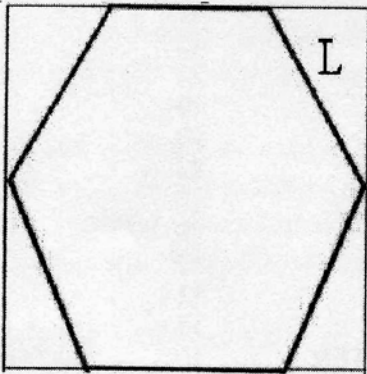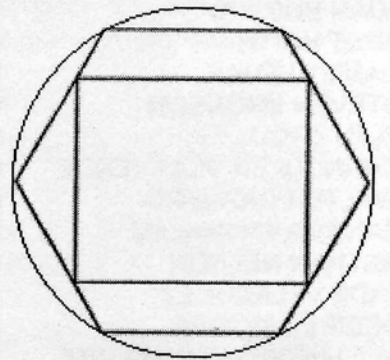

Figure (A)   Figure (B)

B	I	N	G	O
2	17	35	48	61
10	21	39	53	66
14	20	free	55	65
8	25	41	52	62
6	16	37	46	67

3. The Bingo Problem

Each Bingo card (see above) has 24 different numbers selected randomly from the set of integers 1 to 75 inclusive. In one form of the Bingo game numbers are randomly selected from the set 1 through 75 until all the numbers have been selected on the card of some player and that player is then the winner.

(a) ** If 58 numbers are selected at random from the set of integers 1 to 75, find the probability that all 24 numbers on a given card will be included among the 58. Express the answer (**a1**) exactly in terms of factorials and also (**a2**) as a decimal approximation to 4 significant digits. e.g. 0.0003856 or 3.856×10^{-4}.

(b) In (a) if 1000 persons each have one Bingo card, find the probability that at least one of them will have all of the 24 numbers on their card included among the 58 numbers selected. Give the answer to 4 significant digits.

(c) In (b) find the smallest integer N such that if N persons each have one card then the probability is at least 1/2 that one or more persons will have all 24 numbers included among the 58.

** Many scientific calculators only give values for $n!$ if $n \le 69$; however this should not prevent students from getting a solution to (**a2**). Students who are not able to solve (**a2**) may receive partial credit for (b) and (c) by letting A be the answer to (**a2**) and indicating solutions to (b) and (c) in terms of A.

4. The Integer Linear Equation Problem

Given the equation $5x + 7y = c$ we consider, for <u>positive</u> <u>integers</u> c, solution pairs x,y where x and y are <u>non-negative</u> <u>integers</u>.

(**a1**) Show that there is no solution pair x,y if $c = 23$.
(**a2**) Find a solution pair for each of the values of c: 24, 25, 26.

(**b**) Prove that if $c > 23$ then the equation has at least one solution pair.

(**c**) Prove that if c > 69 then the equation has at least two solution pairs.

(**d**) Prove that if a,b ($1 < a < b$) are prime numbers and c is a positive integer, $c \geq ab$, then the equation $ax + by = c$ has at least one solution pair x,y where x,y are non-negative integers.

5. The Difference of Two Squares Problems

In this problem, by a <u>Difference</u> of <u>Squares</u> <u>Representation</u> (DSR) of a positive integer N will be meant a difference of squares of two positive integers which equals N e.g. $3^2 - 2^2$ is a DRS of 5 and $10^2 - 7^2$ is a DSR of 51.

(a1) Find a DSR of 16.
(a2) Show that 14 does not have a DSR.

(**b**) Prove that if N is an odd integer, $N > 1$, then N has at least one DSR.

(**c**) Determine a necessary and sufficient condition that an even positive integer N, $N > 4$, has at least one DSR and prove your answer.

(**d**) Determine with proof a general method for finding all DSR's of a given positive integer N, $N > 4$, and use your method to find all DSR's of 96.