The use of a calculator is permitted only on problem 1. Graphic calculators are not permitted. In order to receive credit for numbers obtained by a calculator it is necessary that numerical expressions used to determine these values be displayed.

1. A Gambling Strategy To Win $1 (A Calculator Problem)

John’s amount of money $A$ is initially $N$ ($N > 1$ is an integer) and he intends to increase his amount to $(N + 1)$ by successive plays of a casino game in which each play has probability $2/5$ of winning and probability $3/5$ of losing the amount bet. His strategy is each time to bet the portion of $A$ to increase the value to $(N + 1)$ or to bet all of $A$ if the value of $A$ is too small to increase the value to $(N + 1)$. Thus for $N = 9$ if $A = 6$ then he would bet $4$ and if $A = 2$ then he would bet $2$. The process ends when either John’s amount $A$ reaches $(N + 1)$ and then he wins, or reaches $0$ and then he loses.

Find to three decimal places the probability John wins if $N =$
(a) 2  (b) 3  (c) 4  (d) 5

(e) Find, with proof, two values of $N, N > 5$, such that John will either win or lose after no more than 4 bets.

(f) Find, with proof, a value of $N$ such that the probability that John wins is greater than 0.99.

2 Quadratic Equations With Roots Between 0 And 1

We consider quadratic equations of the form $ax^2 - bx + c = 0$, where $a, b, c$ are positive integers and the two roots are real numbers in the interval $(0,1)$.

(a) If $a = 6$ and $c = 1$ find, with proof, all values of $b$ such that both roots are in $(0,1)$.

(b) If $a = 5$ find all pairs $(b, c)$ such that both roots are in $(0,1)$.

(c) Given $b^2 - 4ac > 0$ find with proof additional condition(s) relating $a, b, c$ which are necessary and sufficient that both solutions of the equation are real numbers in $(0,1)$. 
3. The Sum of 3 (6) Numbers Divisible by 3 (6) Problem

(a) Give 4 (not necessarily distinct) positive integers such that the sum of any 3 of the 4 is not divisible by 3.

(b) Prove that any set of 5 (not necessarily distinct) positive integers includes 3 whose sum is divisible by 3.

(c) Give 10 (not necessarily distinct) positive integers such that the sum of any 6 of the 10 is not divisible by 6.

(d) Prove that any set of 11 (not necessarily distinct) positive integers includes 6 whose sum is divisible by 6.

4. The Locus of Points Problem

(a) In the given figure, for \( \theta = \pi/4 \), find an equation in terms of \( x, y \) for the locus of points \( P \) for the values \( \alpha, 0 < \alpha < 3\pi/4 \). Sketch the graph of the equation for \( 0 < \alpha < 3\pi/4 \).

(b) As in (a) find the equation, in terms of \( x, y, \theta \), of the locus of points \( P \) for \( \theta \) arbitrary, \( 0 < \theta < \pi/2 \) and \( 0 < \alpha < \pi - \theta \).

(c) In (b) make a sketch and describe the graph of the equation for the case \( \theta \) is a number approximately equal to 0.
5. The Prime Number Problem

In this problem, all number symbols represent positive integers.

Given the equation $1/x - 1/y = 1/n$:

(a) Find all solution pairs $(x,y)$ if $n = 6$; your solution should prove there are no others

(b) Prove that if $n$ is a prime number then there is a unique solution pair $(x,y)$.

(c) Prove that if $n$ has $k$ distinct positive integer divisors, including 1 but not $n$, then there are at least $k$ different solution pairs $(x,y)$.

(d) Find 4 solution pairs if $n = 60$; your work should show how you arrived at your answer.