DAVID ESSNER FINALS XXV 2005-2006

The use of a calculator is permitted only on problem 1. Graphic calculators are *not* permitted. In order to receive credit for numbers obtained by a calculator it is necessary that numerical expressions used to determine these values be displayed.

1. A Gambling Strategy To Win $1 (A Calculator Problem)

John's amount of money \$$A$ is initially \$$N$ ($N > 1$ is an integer) and he intends to increase his amount to \$($N + 1$) by successive plays of a casino game in which each play has probability 2/5 of winning and probability 3/5 of losing the amount bet. His strategy is each time to bet the portion of \$$A$ to increase the value to \$($N + 1$) or to bet all of \$$A$ if the value of A is too small to increase the value to \$($N + 1$). Thus for $N = 9$ if $A = 6$ then he would bet \$4 and if $A = 2$ then he would bet \$2. The process ends when either John's amount A reaches \$($N + 1$) and then he wins, or reaches \$0 and then he loses.

Find to three decimal places the probability John wins if $N =$
(a) 2 (b) 3 (c) 4 (d) 5

(e) Find, with proof, two values of N, $N > 5$, such that John will either win or lose after no more than 4 bets.

(f) Find, with proof, a value of N such that the probability that John wins is greater than 0.99.

2 Quadratic Equations With Roots Between 0 And 1

We consider quadratic equations of the form $ax^2 - bx + c = 0$, where a,b,c are *positive integers* and the two roots are real numbers in the interval (0,1).

(a) If $a = 6$ and $c = 1$ find, with proof, all values of b such that both roots are in (0,1)
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(b) If $a = 5$ find all pairs (b,c) such that both roots are in (0,1).

(c) Given $b^2 - 4ac > 0$ find with proof additional condition(s) relating a,b,c which are necessary and sufficient that both solutions of the equation are real numbers in (0,1).

3. The Sum of 3 (6) Numbers Divisible by 3 (6) Problem

(a) Give 4 (not necessarily distinct) positive integers such that the sum of any 3 of the 4 is not divisible by 3.

(b) Prove that any set of 5 (not necessarily distinct) positive integers includes 3 whose sum is divisible by 3.

(c) Give 10 (not necessarily distinct) positive integers such that the sum of any 6 of the 10 is not divisible by 6

(d) Prove that any set of 11 (not necessarily distinct) positive integers includes 6 whose sum is divisible by 6.

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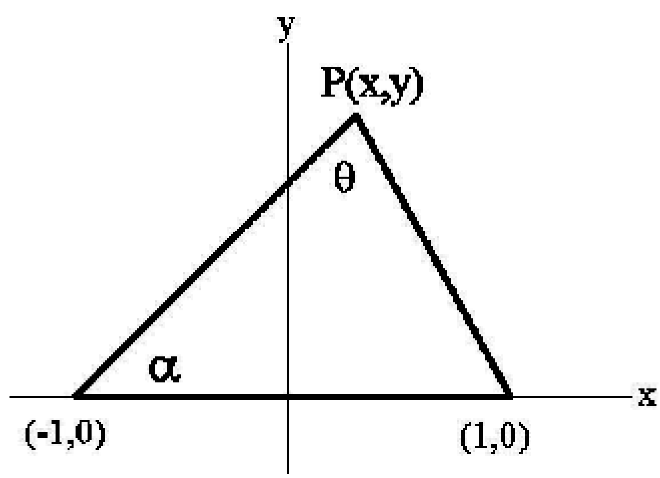
4. The Locus of Points Problem

(a) In the given figure, for $\theta = \pi/4$, find an equation in terms of x,y for the locus of points **P** for the values α, $0 < \alpha < 3\pi/4$. Sketch the graph of the equation for $0 < \alpha < 3\pi/4$.

(b) As in (a) find the equation, in terms of x,y,θ, of the locus of points **P** for θ arbitrary, $0 < \theta < \pi/2$ and $0 < \alpha < \pi - \theta$

(c) In (b) make a sketch and describe the graph of the equation for the case θ is a number approximately equal to 0.



5. The Prime Number Problem

In this problem, all number symbols represent positive integers.

Given the equation $1/x - 1/y = 1/n$:

(a) Find all solution pairs (x,y) if $n = 6$; your solution should prove there are no others

(b) Prove that if n is a prime number then there is a unique solution pair (x,y).

(c) Prove that if n has k distinct positive integer divisors, including 1 but not n, then there are at least k different solution pairs (x,y).

(d) Find 4 solution pairs if $n = 60$; your work should show how you arrived at your answer.