DAVID ESSNER FINALS XXVI 2006-2007

1. The Product of Sums of Numbers and Their Reciprocals Problem

In this problem the symbols x,y,z represent arbitrary **positive real numbers.**

(a) Prove that $x^2 + y^2 \geq 2xy$

(b) Prove that $(x + y)(1/x + 1/y) \geq 4$

(c) Prove that $(x + y + z)(1/x + 1/y + 1/z) \geq 9$

(d) Parts (b) and (c) are special cases of a general inequality. State and prove that inequality.

(e) In (d) describe, with justification, the conditions under which equality holds.

2. Problems of Nearness to the Origin

Given a point P and a set S in the Cartesian plane, then the <u>distance from P to S</u> is the minimum distance from P to a point in S.

(I) Given the right triangle with vertices (0,0), (0,1) and (1,0):

(a) Determine the function (give both equation and domain) $y = f(x)$ which describes the locus curve C of all points which are <u>in</u> (or <u>on</u>) the triangle and are equidistant from the origin (0,0) and the hypotenuse of the triangle.

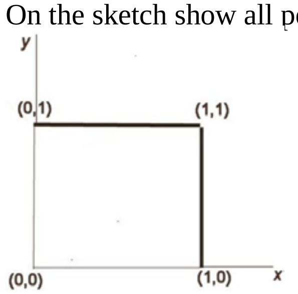
(b) Find the equation of the line segment L which joins the endpoints of the curve C.

(c) Show by a rough sketch the curve C and line L; your sketch should show where C is above or below L.

(II) Given the square with vertices (0,0), (1,0), (0,1) and (1,1):

(a) Determine the function (give both equation(s) and domain(s)) $y = f(x)$ of the curve C which is the locus of points which are <u>in</u> (or <u>on</u>) the square and are equidistant from the origin and the set S consisting of the two sides (including the endpoints) of the square which do not contain the origin.

(b) Show by a sketch the curve C and the set of points in the given square which are closer to the origin than to the set S. On the sketch show all pertinent equations and significant points.



3. The Mod Problem

If *N*,*x*,*y* are non-negative integers then the expression '$N \bmod y \equiv x$' means that $N - x$ is divisible by *y*. In (a)-(c) find the requested values of *N*, showing in each case how these values were determined.

(a1) Excluding $N = 1$ find the smallest positive integer *N*
(a2) Describe all positive integer values for *N*
such that simultaneously $N \bmod 71 \equiv 1$ and $N \bmod 97 \equiv 1$.

(b1) Excluding $N = 1$ find the smallest positive integer *N*
(b2) Describe all positive integer values for *N*
such that simultaneously $N \bmod 12 \equiv 1$, $N \bmod 20 \equiv 1$ and $N \bmod 45 \equiv 1$.

(c1) Find the smallest value of *N*
(c2) Describe all positive integer values for *N*
such that simultaneously $N \bmod 41 \equiv 7$ and $N \bmod 21 \equiv 13$.

4. The Double Transfer Problem

Initially Jar *A* has 1 unit of water and Jar *B* is empty. The portion *r* of Jar *A* is transferred to *B* and then the portion *s* of *B* is transferred to *A*, $0 < r,s < 1$. Thus if $r = ½$ and $s = ¼$ then ½ of the water in *A* is transferred to *B* and then ¼ of the water in *B* is transferred to *A*. This is called a <u>double</u> <u>transference</u> and is to be repeated many times. Let A_n denote the amount of water in *A* after the *n*th double transference.

(a) Find A_1 and A_2 if $r = ½$ and $s = ¼$.

(b) Find a formula for A_{n+1} in terms of A_n, *r*,*s*.

(c) Prove for all values of *r*,*s* ($0 < r,s < 1$) that $A_{n+1} < A_n$ for all positive integer values for *n*.

(d) Find a formula for A_n in terms of *r*,*s*.

(e) If $r = 3/5$ and $s = 1/3$ find integers *a* and *b*, $1 < a < b < 50$ such that A_n is very near a/b whenever *n* is a large integer. Explain your answer.

(f) For *r*,*s* arbitrary find, in terms of *r* and *s*, a numerical expression which approximates A_n for all large *n*.

5. The Sequence of Heads and Tails Problem

A fair (heads and tails equally likely) coin is tossed N times resulting in a sequence of heads (H) and tails (T).

For (a1) $N = 4$
(a2) $N = 10$
find the probability there are two or more successive heads.

(b) For $N = 10$ determine the probability there are three or more successive heads.

(c) For $N = 10$ determine the probability there are two or more successive heads and two or more successive tails.

(d) For $N = 10$ determine the probability there is at least one run of exactly three successive heads.