ESSNER FINALS XXVII 2007-2008

The use of a calculator is permitted only on Problems 2d and 3. In order to receive credit for numbers obtained by a calculator, it is necessary that the numbers used to obtain the calculator result be displayed in the solution.

1. Divisors of 9 and 11

(a) Let N be a four digit positive integer represented by its digits as $abcd$, $a > 0$. Prove N is divisible by 9 if and only if $a + b + c + d$ is divisible by 9.

(b1) Prove if n is an odd positive integer then $10^n + 1$ is divisible by 11

(b2) For the case n is an even positive integer, determine without proof a formula for the integer, closest to 10^n, that is divisible by 11.

(b3) Let M be a four digit positive integer represented by its digits as $abcd$, $a > 0$. Prove M is divisible by 11 if and only if $-a + b - c + d$ is divisible by 11.

2. The Pythagorean Integer Triple Problem

All numbers to be considered in this problem are positive integers. A Pythagorean Triple (**PT**) is an ordered set of three numbers (a,b,c) such that $a^2 + b^2 = c^2$; e.g. (3,4,5) and (4,3,5) are each a **PT** but (5,3,4) is not a **PT**. .

(a) Prove that if (a,b,c) is a **PT** and n is a positive integer then (na,nb,nc) is a **PT.**

(b) Prove that if p, $p > 2$, is a prime number then there are integers b,c such that (p,b,c) is a **PT**.

(c) Prove that if n, $n > 2$, is a positive integer then there are integers b,c such that (n,b,c) is a **PT**.

(d) (A calculator is permitted) Find two prime numbers p, $p > 100$, for which there exist numbers a,b such that (a,b,p) is a **PT**. Show how you arrived at your values for p.

3. Missile Problem (A calculator problem)

(a) An object moves counterclockwise at constant speed in a circular path of radius 60 miles, making 1 complete revolution each hour (see the figure below). When the object is at position ***A***, a missile is fired from the center ***O*** of the circle at a constant speed of 600 mph. If the missile hits the object at point ***P***:

(a1) Determine the time it takes the missile to reach ***P***.

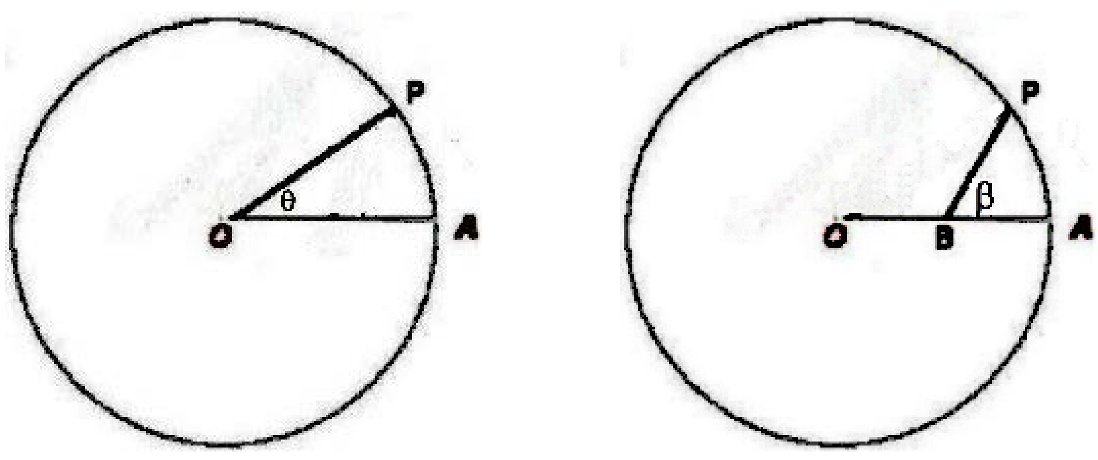
(a2) Determine θ, where θ is the angle measured in degrees counterclockwise from ***OA*** to ***OP***;

(b) Let the point ***B*** be 30 miles from ***O***, collinear with the segment ***OA*** and such that ***B*** is between ***O*** and ***A*** (as in the figure below). If the missile is fired from ***B***, when the object is at ***A***, and hits the object at ***P***:

(b1) Let ***T*** be the time it takes the missile to hit the object at ***P*** and β the angle measured counterclockwise from ***BA*** to ***BP.*** Find a formula for β in terms of ***T.***

(b2) Obtain an equation in which ***T*** is the only unknown.

(b3) Determine the value of ***T*** to the nearest 0.1 minute, and the value of β to the nearest 0.1 degree.



(a) **(b)**

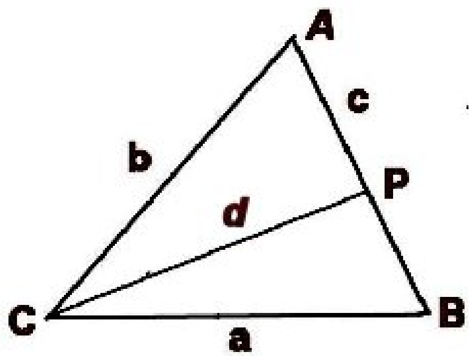
4. The Triangle Problem

Let *ABC* be an acute triangle with vertices *A*,*B*,*C* and corresponding opposite sides of lengths *a*,*b*,*c*. (See the figure below). Let *P* be a point on the side *AB*, and *d* the length of the segment *CP*.

(a) If *P* is the midpoint of the side *AB* find a formula for the measure of angle *C* (= $\angle ACB$) in terms of *a*,*b*,and *d*. If convenient you may express the formula as a solution for sin *C*, cos *C* or tan *C* instead of *C*.

(b) If *CP* bisects angle *C*, find a formula for the measure of angle *C* (= $\angle ACB$) in terms of *a*,*b*,and *d*. If convenient you may express the formula as a solution for sin *C*, cos *C* or tan *C* instead of *C*.

(Note: You may find it helpful to use a Cartesian (xy) coordinate system in parts (a),(b)).



5 The Bug Random Walk Problem

(a) A bug walks around a triangle with vertices *A*,*B* and *C*; at each vertex it moves to another vertex with equal probability. If it starts at vertex *A* then

(a1) Determine the probability it is at vertex *A* after 1,2,3,4,5 and 6 moves.

(a2) If p_n is the probability it is at position *A* after *n* moves, find a formula for p_{n+1} in terms of p_n; give a proof for your formula.

(a3) Find a formula for p_n *in* terms of *n*, $n > 1$; give a proof for your formula.

(a4) Determine the probability that, neither after the third move nor after the seventh move, the bug is at vertex C.

(a5) Determine the probability the bug returns to A at least twice in its first 12 moves.