DAVID ESSNER FINALS XXVIII 2008-2009

1. Cubic Polynomial Roots Problems

Given the equation $x^3 + Ax^2 + Bx + C = 0$, $ABC \neq 0$.

(a) If $x = 1$ is a root show that $B + C$ equals the sum of the other two (possibly complex) roots.

(b) If all the roots are real, prove that $A^2 > 2B$.

(c) If $C = 1$, A,B are integers and $A + B$ is odd, prove there are no integer roots.

2. The Sum of Reciprocals Problem

In this problem all number symbols represent positive integers.

(a) Find a value for m,n such that $4/11 = 1/m + 1/n$.

(b) Prove that there do not exist numbers m,n such that $5/11 = 1/m + 1/n$.

(c) Let p be a prime number and $1 < a < p$. Prove that there exist m,n such that $a/p = 1/m + 1/n$ if and only if a divides $(p + 1)$.

<u>*Note:*</u> You may assume the conclusion of (c) to solve (d) even if you were not able to prove (c).

(d) From the conclusion of (c) determine all values of a, $a < 17$, for which there exist m,n such that $a/17 = 1/m + 1/n$; in each case gives values for m and n.

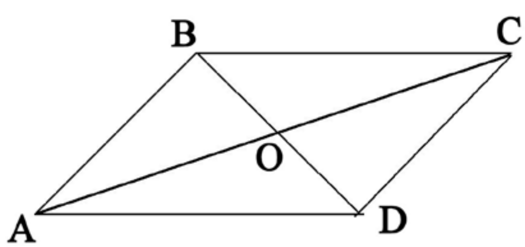
3. Parallelogram Problem

(See the (not to scale) figure below)

Given a parallelogram $ABCD$, let O be the intersection of the diagonals :

(a) If $AB = 2$, $BD = 6$ and $\angle AOB = 30°$, find to two decimal places all possible values for BC.

(b) If $AC = 6$, $BD = 4$ and $\angle BAD = 60°$, find all possible <u>exact</u> values for AB and BC.



4. An Approximation to $\sqrt{2}$ Problem

In this problem we consider the sequence of rational numbers (quotients of integers) x_1, $x_2,\ldots,x_n,\ldots$ where $x_1 = 1/1$ and if $x_n = r_n/s_n$ then $x_{n+1}=(r_n + 2s_n)/(r_n + s_n)$. Thus for example $x_2 = (1 + 2*1)/(1 + 1) = 3/2$ and $x_3 = (3 + 2*2)/(3 + 2) = 7/5$.

(a) Find x_4 and x_5, and compute to four decimal places the values $x_n^2 - 2$ for $n = 1,2,3,4,5$.

(b) Explain informally why $x_n \geq 1$ for all n.

(c) Find a formula for x_{n+1} in terms of x_n.

(d) Prove that for each positive integer n: $x_{n+1}^2 - 2 = (2 - x_n^2)/(1 + x_n)^2$
(Thus the x_n^2 alternate between being less than and greater than 2).

(e) Prove that for each positive integer n: $|x_{n+1}^2 - 2| \leq |x_n^2 - 2|/4$.
(Thus as n increases the values of x_n^2 become closer to 2).

(f) Prove that for each positive integer n: $|x_n^2 - 2| \leq 1/4^{n-1}$.
(Thus for increasing n the values of x_n^2 become arbitrarily close to 2).

(g) Prove that for each positive integer n: $|x_n - \sqrt{2}| \leq 1/2^{2n-1}$
(Thus for increasing n the x_n's become arbitrarily close to $\sqrt{2}$)

5. The TV Game Show $n \geq 4$ Curtain Problem

In a TV game show there are four curtains numbered *C*1,*C*2,*C3*,*C4*. There is a prize behind one curtain (to be denoted the *prize curtain*) and nothing behind the other three curtains. Initially it is equally likely for the prize to be behind each of the four curtains. A contestant initially guesses that the prize is behind *C*1. The host then opens a curtain other than the prize curtain, and then offers the contestant an opportunity to change the guess. The contestant then makes a final selection of a curtain different from *C*1 and different from the opened curtain. Find the probability there is a prize behind the final selected curtain if the selection by the host of the curtain to be opened is

(a) random (equally likely) among all curtains other than the prize curtain.
(b) random among all curtains other than the prize curtain and *C*1.

In parts (c) and (d) below it will be assumed the TV game show has n, $n \geq 4$, curtains C1,C2,…,C*n* . The host will open r, $1 \leq r \leq n - 2$, different curtains and after each opening the contestant will change to a selection which is different from the previously opened curtains and the current selection (but might change to a selection which occurred previous to the current selection). The symbols $p(n,r\text{-}1)$ and $p(n,r)$ will respectively denote the probability the selection by the contestant, after r-1 and r curtains have been opened, is the prize curtain.

(c) (Extension of (b)). Assume each curtain which is opened by the host is chosen at random among all curtains other than the previously opened curtains, the prize curtain and the current selection by the contestant.
 (c1) If $p(n,r\text{-}1) = t$, find a formula for $p(n,r)$ in terms of t, r, and n.
 (c2) Determine $p(4,2)$ and p(5,2). You must show your work or explain your answer.

(d) (Extension of (a)). Assume each curtain which is opened by the host is chosen at random among all curtains other than the previously opened curtains and the prize curtain.
 (d1) Determine $p(5,2)$ and $p(5,3)$. You must show your work or explain your answer.
 (d2) From observation of the values in (d1) state and prove a formula for $p(n,r)$ in terms of n,r.