DAVID ESSNER FINALS XXIX 2009-2010

1. Oddity of Pairs of Success Fraction Sequences

This problem concerns the comparison of success fraction sequences of A and B over each of two time periods and the combined time periods. As an example: A has success 3 times in 4 tries in period 1 and 3 times in 5 tries in period 2; the sequence for A is then A = [3/4, 3/5, 6/9]: B has success 23 times in 32 tries in period 1 and 2 times in 5 tries in period 2; the sequence for B is then B = [23/32, 2/5, 25/37]. In this example A's success fraction was greater than B's in periods 1 and 2, but B's success fraction in the combined periods was greater than A's. This situation is called an ***Oddity***.

Hereafter in this problem A and B will denote success fraction sequences; each fraction is greater than 0. The pair (A,B) is an ***Oddity*** if A's success fraction is greater than B's in both of periods 1 and 2, but B's success fraction is greater than A's in the combined period.

(a) Given A = [1/2, 2/3, 3/5] and B = [1/3, $(2x – 1)/3x$, $2x/(3x + 3)$] determine the values of x for which (A,B) is an Oddity.

(b) If A = [4/7, 5/9, 9/16] find a B so that (A,B) is an Oddity.

(c) If A = [a/b, c/d, $(a + c)/(b + d)$] prove that if $a/b = c/d$ then there is no B such that (A,B) is an Oddity.

(d) If A = [a/b, c/d, $(a + c)/(b + d)$] and $a/b > c/d$ prove that there is a B such that (A,B) is an Oddity.

2. The Reciprocals Triple Problem

In this problem x,y,z will denote positive integers where $x > 1$, $y > 1$, $z \geq 1$ and $1/x + 1/y = 1/z$. An example of an (x,y,z) triple for this equation is (2,2,1) since 1/2 + 1/2= 1/1.

(a) For (a1) $x = 3$ (a2) $x = 7$ find a triple (x,y,z) such that $1/x + 1/y = 1/z$.

(b) Determine for each of (b1) $x = 6$ (b2) $x = 9$ two triples (x,y,z) such that $1/x +1/y = 1/z$.

(c) Prove that if x is a prime number then there is a unique triple (x,y,z) such that $1/x + 1/y = 1/z$.

(d) Prove that if x is not a prime number then there are at least two triples (x,y,z) such that $1/x + 1/y = 1/z$.

3. The Unique Intersection Parabolas Problem

Given that parabolas $y_1(x) = a_1x^2 + b_1x + c_1$ and $y_2(x) = a_2x^2 + b_2x + c_2$ have exactly one point of intersection, at $x = 0$:

(a) If $a_2 > a_1$ prove that $y_2(x) > y_1(x)$ for all $x \neq 0$.

(b) If $a_2 = a_1$ prove that $y_2(x) > y_1(x)$ for some values of x and $y_2(x) < y_1(x)$ for some values of x.

(c) If $a_2 > a_1$ give an example in which the point of intersection
  (c1) is the vertex of both parabolas.
  (c2) is not the vertex of either parabola.

(d) If $a_2 > a_1$ prove the vertex of $y_2(x)$ occurs at $x = 0$ if and only if the vertex of $y_1(x)$ occurs at $x = 0$.

4. The Two Operations Accessible Problem

Two operations: C (*commute*), A (*addition*) on the set of pairs of positive integers are to be defined by: $C(x,y) = (y,x)$; $A(x,y) = (x, x + y)$. There is also an *initial point* I = (1,1). *Strings* of C's and A's, applied from right to left, will denote successive applications of C and A. For example AACA(I) = AAC(1,2) = AA(2,1) = A(2,3) = (2,5). No string will have two successive C's since $CC(x,y) = (x,y)$. A pair (x,y) will be called *accessible* if there is a string S such that $S(I) = (x,y)$. Thus, from the above, (2,5) is accessible. Two positive integers are *relatively prime* (rp) if their greatest common divisor is 1.

(a) Find ACACAAA(I)

(b) Prove that if x and y are rp then so are (b1) x and $x + y$ (b2) x and $y - x$ if $y > x$.

(c) Prove that if (x,y) is accessible then x and y are rp.

(d) Prove, or use an algorithm to show, that if x and y are rp then (x,y) is accessible.

(e) Find, showing your work, a string S such that S(I) = (12,19). Is this string unique? Explain

5. The Competition Problem

A and B play a series of points. Each point either A or B goes first; one of them is the winner and one is added to the score of the winner. The loser of a point goes first in the next point. If A goes first then the probability that A wins the point is r, $0 < r < 1$; if B goes first then the probability B that wins the point is s, $0 < s < 1$. It will be assumed that A initially goes first. The symbol p_n will denote the probability that A goes first on the n^{th} point; thus $p_1 = 1$.

(a) Find p_3 in terms of r and s.

(b) Find a formula for p_{n+1} in terms of p_n, n = 1,2, … .

(c) Find a formula for p_n in terms of n. Express the answer in closed form; i.e. the number of arithmetic operations in the answer should not depend on n.

Note: The solutions to parts (d)-(f) are independent of the solutions or conclusions of parts (a)-(c).

(d) In terms of r and s find the probability t that A wins two points before B wins two points.

(e) Find: (e1) A formula for s in terms of r if A and B have equal probability of winning the first two points.
(e2) The range of r values for which the formula in (e1) is valid i.e. for which the value of s is between 0 and 1.

(f) A and B play a best 3 of 5 series (the series ends when one player has scored three points and is then the winner). If $s = r$ and t is the probability calculated in part (d), find in terms of r and t an expression for the probability that A wins three points before B wins three points.