# DAVID ESSNER FINALS XXX 2010-2011

*20 points*

(1) **Sequences Whose Successive Numbers Differ By One**
In this problem an $n$–*sequence* will mean a sequence $a_1, a_2, \ldots, a_n$ of $n$ non-negative integers where $a_1 = 0$ and $|a_i - a_{i-1}| = 1$ for each $i$, $2 \le i \le n$. Thus $0, 1, 2, 1, 0, 1, 2, 3$ is an example of an 8–sequence. The sum of an $n$–sequence is the sum of its elements; thus 10 is the sum of the above example.

(a) List all (a1) 4–sequences and (a2) 5–sequences.

(b) Let $L(n)$ be the largest possible sum for an $n$–sequence and $S(n)$ be the smallest possible sum for an $n$–sequence.
  (b1) Determine a formula for $L(n)$ in terms of $n$.
  (b2) Determine a formula for $S(n)$ in terms of $n$.

(c) Find three 12–sequences whose sums are 8, 10, and 20 respectively.

(d1) For each $n$, prove that either the sum of every $n$–sequence is even or the sum of every $n$–sequence is odd. For which $n$ are these sums even? For which $n$ are these sums odd?

(d2) For each $n$, determine the set of possible sums of $n$–sequences and justify your answer.

*21 points*

(2) **The Basketball Combinations Problem**

(a) In basketball in 1950 a player could score in two ways:

(i) 2 points for a field goal
(ii) 1 point for a free throw

Let $S_n$ be the (unordered) number of ways a player could score $n$ points. For example $S_5 = 3$ from the cases: $\{2, 2, 1\}$, $\{2, 1, 1, 1\}$, $\{1, 1, 1, 1, 1\}$

(a1) For $n = 6$ list the ways to score 6 points and determine $S_6$.

(a2) For $n$ an arbitrary positive integer determine a formula for $S_n$ in terms of $n$; explain your solution.

(b) In the present day a basketball player can score points in three ways:

(i) 3 points for a long field goal
(ii) 2 points for a regular field goal
(iii) 1 point for a free throw

As in (a) let $T_n$ be the number of (unordered) ways a player could score $n$ points. For example $T_4 = 4$ since there are four ways: $\{3, 1\}$, $\{2, 2\}$, $\{2, 1, 1\}$, $\{1, 1, 1, 1\}$

(b1) Find $T_3$, $T_5$, $T_6$; in each case list the unordered ways.

(b2) Find $T_{12}$, $T_{18}$, and $T_{30}$.

(b3) Find an explicit formula for $T_n$ when $n = 6k$ for a positive integer $k$.

*22 points*

(3) **The Right Triangle Problem**

Let $ABC$ denote a right triangle with hypotenuse $AB$, and let $E$, $F$ be points on $AB$ such that $E$ lies between $A$ and $F$, $|CF| = 2$ and $|CE| = d$, $d > 0$. (See the figure below, which is not to scale for any parts of this problem.)

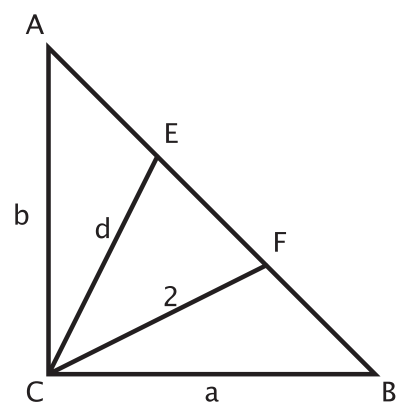
(a) Given $\angle FCB = 30°$ and $\angle ECF = 45°$, determine the interval of possible values of $d$. Evaluate the endpoints of the interval to two decimal places.

(b) Now assume the segments of the hypotenuse have the proportion $|AE| : |EF| : |FB| = 1 : 2 : 3$.

(b1) Give an equation showing the relationship between $a$ and $b$.

(b2) Give a formula for $d$ in terms of $a$.

(b3) Determine the interval of possible values of $d$.



*23 points*

(4) **The Greatest Common Divisor Problem**

In this problem $a$, $b$ will denote positive integers, and $d$ will denote the greatest common divisor (gcd) of $a+b$ and $a^2+b^2$.

(a) For each (a1) $d=2$, (a2) $d=3$, and (a3) $d=5$ give an example of a pair of positive integers $a, b$ such that $d$ is the gcd of $a+b$ and $a^2+b^2$. Explain your answers.

(b) Prove that if $b=2$ and $a$ is an odd integer then $d=1$.

(c) Show that for infinitely many positive integers $d$ there is a pair of positive integers $a, b$ such that $d$ is the gcd of $a+b$ and $a^2+b^2$.

(d) Show that for each $d$, $d>1$, there are infinitely many pairs of positive integers $a, b$ such that $d$ is the gcd of $a+b$ and $a^2+b^2$.

*25 points*

(5) **The Urn Problem**

There were $N$ green urns and $N$ yellow urns. In each urn were $N$ red balls and $N$ black balls.

A total of $N$ balls had been drawn from the urns as follows:
Step 1: An urn was selected at random. A ball was drawn from it and then returned to the same urn.
Step 2: Step 1 was repeated $N-1$ more times.

(a) Suppose you are told that among all the drawn balls, at least one was red. Let $P(N)$ be the probability that more than one of the drawn balls was red.
   (a1) What is $P(2)$?
   (a2) What is $P(3)$?
   (a3) What is $P(N)$ for $N \geq 2$?

(b) Suppose you are told that among all the drawn balls, at least one was both red and selected from a green urn. Let $Q(N)$ be the probability that more than one of the drawn balls was red.
   (b1) What is $Q(2)$?
   (b2) What is $Q(3)$?
   (b3) What is $Q(N)$ for $N \geq 2$?

(c) Suppose you are told that among all the drawn balls, at least one was red. Let $R(N)$ be the probability that more than half of the drawn balls was red. What is $R(N)$ for $N \geq 2$?