DAVID ESSNER FINAL EXAM
XXXI 2011-2012

22 points (1) **Dividing One Less Than a Square**

In this problem x, n will denote positive integers. We say n *divides* $x^2 - 1$ if $(x^2-1)/n$ is a non-negative integer.

(a) Prove that if either $x = 15k+1, 15k+4, 15k+11$, or $15k+14$ for some non-negative integer k, then 15 divides $x^2 - 1$.

(b) Prove that 7 divides $x^2 - 1$ if and only if either $x = 7k+1$ or $x = 7k+6$ for some non-negative integer k.

(c) Prove that if k is a non-negative integer, then n divides $(kn+x)^2 - 1$ if and only if n divides $x^2 - 1$.

(d) Describe all positive integers x for which 21 divides $x^2 - 1$.

21 points (2) **2012**

A *ternary string of length* n is a sequence $x_1x_2x_3 \dots x_{n-1}x_n$ where each term x_i is 0,1, or 2. We say 2012 is a *substring* if $x_i = 2$, $x_{i+1} = 0$, $x_{i+2} = 1$, and $x_{i+3} = 2$ for some i. For example, 12212012 is a ternary string of length 8 that ends with the substring 2012 while 22010211 is a ternary string of length 8 which does not contain 2012 as a substring.

(a) How many ternary strings of length 8 neither start with the substring 2012 nor end with the substring 2012?

(b) How many ternary strings of length 8 do not contain 2012 as a substring?

(c) How many ternary strings of length n have either no 0's or no 1's or at most one 2?

(d) How many ternary strings of length 2012 have the property that, other than x_{2012}, every term that is a 2 is the first term of a substring 2012.

23 points (3) **The Functional Equation Problem**

Let f be a real-valued function of one variable that satisfies the equation

$$(*) \qquad [f(x+y)]^2 + [f(x)]^2 + [f(y)]^2 - 2f(x+y)f(x)f(y) = 1$$

for all real numbers x and y.

(a) Show that $f(0)$ must equal either 1 or $-1/2$.

(b) Suppose $f(0) > 0$. Prove that $f(x) = f(-x)$ for all x.

(c) Suppose for some $a > 0$ that $f(a) = 0$ and $f(x) > 0$ for all $0 \leq x < a$. Determine the values of

(i) $f(a/2)$, (ii) $f(a/3)$ and $f(2a/3)$.

(d) Suppose $f(0) < 0$. Prove that f must be a constant function.

(e) Verify that the following functions satisfy equation $(*)$ above.

(i) $f(x) = \cos(x)$ (ii) $f(x) = \dfrac{2^x + 2^{-x}}{2}$

21 points

(4) Drawing Red Balls

An urn has $2n$ balls of which k, $1 \leq k \leq n$, are red. Players A and B take turns drawing a ball (without replacement) at random from the urn until all k red balls have been drawn. Assume that Player A goes first.

(a) (i) In the case $n = 5$ and $k = 1$, calculate the probability that Player A draws the red ball.
(ii) In the case $n = 5$ and $k = 2$, calculate the probability that Player A draws the last red ball.
(iii) In the case $n = 5$ and $k = 2$, calculate the probability that Player A draws both red balls.

(b) For general n and k, calculate the probability that Player A draws all the red balls.

(c) In the case when $k = 2$, calculate the probability that Player A draws the last red ball.

(d) In the case when $k = 3$, determine which of the players, if either, has the greater chance of drawing the last red ball.

24 points

(5) The Tetrahedron Problem

The four faces of a regular tetrahedron, shown below, are all equilateral triangles with length 1 edges. Let T be the union of these triangles and let S be the enclosed solid.

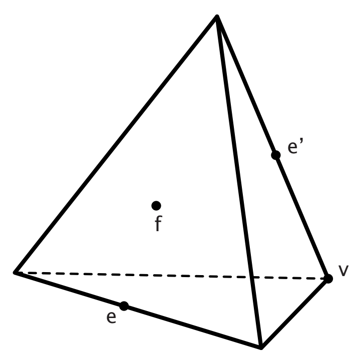
The S-*distance* between two points a and b in S is the length of the straight line segment connecting the two points.

The T-*distance* between two points a and b in T is the shortest length of the paths contained in T connecting the two points.

A *path in* T connecting points a and b is a finite sequence of line segments

$$\overline{P_0P_1}, \overline{P_1P_2}, \overline{P_2P_3}, \ldots, \overline{P_{n-1}P_n}$$

where $a = P_0$, $b = P_n$, and the terminal point of one segment is the initial point of the next. The *length* of a path is the sum of the lengths of the constituent line segments.



(a) Let v be a vertex of T and let f be the center of the opposite face as shown in the figure.
  (i) Show that the S-distance between v and f is $\sqrt{6}/3$.
  (ii) Show that the T-distance between v and f is $2\sqrt{3}/3$.

(b) Let e and e' be midpoints of a pair of opposite edges of T as shown in the figure.
  (i) What is the S-distance between e and e'?
  (ii) What is the T-distance between e and e'?

The T-*disk* of radius r centered at a point c in the tetrahedron T, denoted $D_T(c, r)$, is the subset of points in T of T-distance r or less from c.

(c) Find the areas of the following T-disks of radius $\frac{1}{2}$.
  (i) $D_T(v, \frac{1}{2})$
  (ii) $D_T(e, \frac{1}{2})$

(d) Find the areas of the following T-disks of radius 1.
  (i) $D_T(v, 1)$
  (ii) $D_T(e, 1)$