DAVID ESSNER FINAL EXAM

XXXII 2012 – 2013

20 points (1) **The Acute Triangle Altitudes Problem**

(a) Is there an acute triangle so that the altitudes to its three sides have lengths $1, 1, 2$? Either determine the lengths of the sides of such a triangle or prove it cannot exist.
(b) Is there an acute triangle so that the altitudes to its three sides have lengths $1, 2, 2$? Either determine the lengths of the sides of such a triangle or prove it cannot exist.
(c) If two of the altitudes of an acute triangle have lengths $1, 2$ determine the possible values of the length of the other altitude.

20 points (2) **Expressions with Ones and Pluses**

A finite string of the two symbols `1` and `+` is an *arithmetic expression* if the symbol `+` neither begins nor ends the string and no two `+` symbols are adjacent. The number of symbols in a string is its *length*. If the string is an arithmetic expression, it can be evaluated to give an integer. For example the string `11+1+111+1` is an arithmetic expression of length 10 with value 124, and the string `1111` is an arithmetic expression of length 4 with value 1111. The strings `111+` and `11++1+111` are not arithmetic expressions.

(a) For each positive integer n, how many arithmetic expressions are there with exactly n `1`'s?
(b) How many arithmetic expressions of length 7 are there?
(c) For each positive integer n, how many arithmetic expressions are there of length n?
(d) Among the arithmetic expressions of length 13 with 3 `+`'s, how many distinct values are there?
(e) Among the arithmetic expressions of length 13, how many distinct values are there?

20 points (3) **Product-Difference of Prime Numbers**

In this problem a number d is a *Product-Difference of Prime Numbers (PDP)* if there are prime numbers p, q with $0 < p < q$, such that $d = pq - p - q$. Integers p, q which satisfy this equality are representations of d. For example the integers $2, 3$ give a representation of 1 as a PDP since $1 = 2 \times 3 - 2 - 3$.

(a) Prove that every PDP is a positive odd integer.
(b) What is the smallest positive odd integer that is not a PDP?
(c) Assume k is an integer. Prove that $4k + 1$ is a PDP if and only if $4k + 3$ is prime.
(d) Find a positive integer that has exactly four different representations as a PDP?

22 points (4) **Shuffle Functions and Unstable Orbits**

Let f be any real-valued function defined on a subset D of real numbers. Say that f *is a shuffle* if there exists some real number $c > 0$ such that

$$|f(x) - x| > c \text{ for all } x \text{ in } D.$$

For any x_0 in D, recursively define $x_n = f(x_{n-1})$ for all positive integers n for which x_{n-1} is in D. We say that x_0 *leads to an unstable orbit* if either x_n is not in D for some positive integer n, or there exists some real number $d > 0$ such that

$$|x_{n+1} - x_n| > d \text{ for all } n \geq 1.$$

Consider now the function f defined on the set D of all nonzero real numbers by

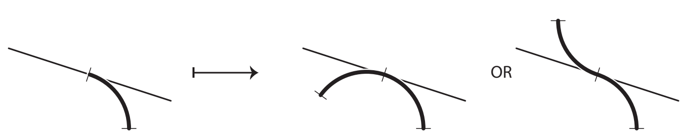
$$f(x) = \frac{1}{2}\left(\frac{a}{x} + x\right),$$

where a is some fixed real number.

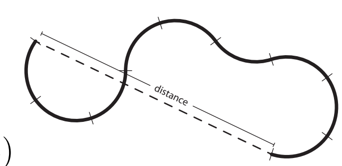
(a) Prove that f is a shuffle if and only if $a < 0$.
(b) When $a > 0$, prove that there exists some real number $b > 0$ such that for all $x > 0$,
  (i) $f(x) - b = \frac{1}{2x}(x - b)^2$, and
  (ii) $0 \leq f(f(x)) - b \leq \frac{1}{2}(f(x) - b)$.
(c) For each real number a, determine the values of x_0 for which x_0 leads to an unstable orbit.

22 points (5) **The Train Track Problem**

For an integer $n \geq 2$, an *n–train track* is a path in the plane formed from arcs of a unit circle of angle $2\pi/n$ arranged so that consecutive arcs are tangent to the same line. Figure (A) shows the two ways that one arc may be continued by another together with their common tangent line. We allow the path to intersect itself.



(A)



(B)

The *length* of an n–train track is the number of arcs used. The *distance* of an n–train track is the distance in the plane between its beginning and end. Figure (B) shows an example of a 5–train track of length 9. Its distance is the length of the indicated line segment.

(a) What is the greatest distance of an n–train track of length n?
(b) What is the set of distances of the 4–train tracks of length at most 4?
(c) What is the set of distances in the interval $[0, 3]$ that may be obtained by a 6–train track?
(d) What is the shortest distance among 8–train tracks with length 9?