DAVID ESSNER MATHEMATICS COMPETITION XXXV

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(1) *16 points* **The Sequence Pair Period Problem**

Let j, k be positive integers with $j < k$. The sequence $S_{j,k} = \{(a_0, b_0), (a_1, b_1), (a_2, b_2), \dots\}$ of ordered pairs of integers satisfies the following conditions:

- $(a_0, b_0) = (0, 0)$
- If $a_n \geq b_n$ then $a_{n+1} = a_n$ and $b_{n+1} = b_n + j$.
- If $a_n < b_n$ then $a_{n+1} = a_n + k$ and $b_{n+1} = b_n$.

The *period* $P(j, k)$ of the sequence $S_{j,k}$ is the least positive integer N such that $a_N = b_N$.

(a) For the sequence $S_{3,5}$:
  (i) Describe the values of n such that $a_n = b_n$.
  (ii) Determine a_{100}.

(b) If k is a multiple of j, express $P(j, k)$ in terms of j and k.

(c) Determine $P(2, k)$ in terms of k.

(d) Show that if m is a positive integer then $P(mj, mk) = P(j, k)$.

(e) If j is an even integer and $k = j + 2$ determine $P(j, k)$ in terms of j.

(f) If j and k are relatively prime show that $P(j, k) = j + k$.

(g) Determine $P(84, 105)$.

(2) *21 points* **The Sum with the Digit Sum Problem**

A positive integer N is *proper* if there is a positive integer n such that $N = n + S(n)$ where $S(n)$ is the sum of the (decimal) digits of n. If $N = n + S(n)$ then we say n *supports* N. A positive integer N is *improper* if no positive integer n supports N.

(a) (i) Show that if N is a proper integer with $N < 100$, then there is a unique integer n that supports N.
  (ii) Prove that if $10 < N < 89$, then N is proper if and only if $N + 11$ is proper.
  (iii) List the positive integers less than 100 that are improper.

(b) What is the smallest proper integer that is supported by two different integers?

(c) List all the proper integers $N < 1000$ for which $N + 11$ is not proper.

(d) Can a proper integer be supported by three positive integers? If not, why not? If so, what is the smallest such proper integer?

(3) *20 points* **The After-Before Functions Problem**

For fixed real numbers a and b, consider the set $\mathcal{F}$ of all real valued functions f defined on the real line that satisfy

$$f(x) = af(x + 1) + bf(x - 1)$$

for all real numbers x.

We say such a function f is *bounded* if there exists a number M such that $|f(x)| \leq M$ for every real number x.

(a) When $a = b = 1$, either find a function in $\mathcal{F}$ that is non-constant and satisfies $f(x) \geq 0$ for all real numbers x or show no such function exists.

(b) When $a = b = 1$, either find a function in $\mathcal{F}$ that is non-constant and bounded or show no such function exists.

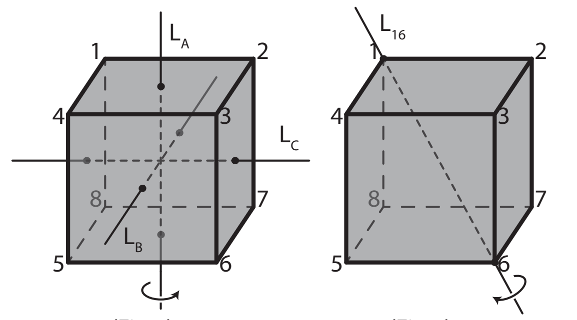
(c) When $a = b = \frac{1}{2}$, show that there are infinitely many functions in $\mathcal{F}$ that satisfy $f(x) > f(y)$ whenever $x > y$.

(d) When $a = b = \frac{1}{2}$, either find a function in $\mathcal{F}$ that is bounded and $f(x) \neq f(x + 1)$ for some real number x or show no such function exists.

(e) When $a = \frac{1}{4}$ and $b = \frac{3}{4}$, either find a function in $\mathcal{F}$ that is not bounded or show no such function exists.

(4) *21 points* **The Rotating Cube Problem**

Consider a cube with vertices labeled with the numbers 1 through 8 and the seven following lines through the cube. The lines passing through the centers of opposite faces will be denoted by L_A, L_B and L_C as shown in Figure 1. The lines joining vertices 1 and 6, 2 and 5, 3 and 8, 4 and 7 will be denoted respectively by L_{16}, L_{25}, L_{38}, L_{47}. Line L_{16} is shown in Figure 2.



(Fig 1) (Fig 2)

In this problem, we will be rotating the cube about these seven lines by certain angles, and these lines will move with the cube. Let us say a rotation is a *cube-rotation* if the set of vertices is preserved as a whole, though individually the vertices may change positions. For example, the rotation of the cube about the line L_A by 90° in the direction indicated in Figure 1 is a cube-rotation, and the line L_B moves to where the line L_C was.

(Two rotations that differ by a full rotation will be considered equivalent.)

(a) (i) What angles of rotation about the line L_A give cube-rotations? Use the direction given in Figure 1.

(ii) For each cube rotation about L_A, make a table to record the changes of the vertices.

(E.g

1	2	3	4	5	6	7	8
8							

indicates vertex 1 moved to where vertex 8 was.)

(b) (i) What angles of rotation about the line L_{16} give cube-rotations? Use the direction given in Figure 2.

(ii) For each cube rotation about L_{16}, make a table to record the changes of the vertices.

(c) Consider the cube-rotation about the line L_{16} by 120° in the direction indicated in Figure 2 and the resulting positions of the vertices.

(i) Among the six other lines L_A, L_B, L_C, L_{25}, L_{38}, and L_{47}, find a pair of lines so that a rotation around the first line followed by a rotation around the second line will result in the vertices of the cube being in the same position as the 120° cube-rotation about the line L_{16}. Is this pair unique?

(ii) Must the two rotations in the previous problem be cube-rotations?

(d) Is there a sequence of cube-rotations for which we have the following partial table of vertex changes?

1	2	3	4	5	6	7	8
7				5		1	

(e) If for each finite sequence of cube-rotations we record the resulting changes of vertices in a table, show there are at most 24 different tables obtained.

(5) *21 points* **The Folding Paper Problem**

A square sheet of paper lies flat on a table. The paper is perfectly folded in half once, keeping half of the paper in place on the table, to make a rectangle. This new rectangle is two paper layers thick. This new rectangle is then folded in half again, keeping half of the rectangle in place on the table, to make another rectangle. This rectangle is now four paper layers thick. Observe that after these two folds, the bottom layer of paper is a rectangle that has not moved from its original position in the original square. We say such a rectangle is obtained from the two folds.

For the rest of this problem, by *fold* we mean a perfect fold in half where one half is not moved as in the description above.

(a) What rectangles in the original square can be obtained as the bottom layer of some sequence of two folds?

(b) For each of the rectangles in part (a), how many ways can the original square be folded to obtain that rectangle?

(c) Consider dividing the original square into a 16×16 grid of smaller squares.

(i) How many folds are needed to obtain one of these smaller squares?

(ii) Show that each of these smaller squares can be obtained by a sequence of folds.

(iii) What is the maximum number of ways a rectangle formed by 2 of these smaller squares may be obtained by a sequence of folds?

(iv) Which squares formed by 4 of these smaller squares can be obtained by a sequence of folds?

(d) For a given positive integer N, consider dividing the original square into a $2^N \times 2^N$ grid of squares. Among the rectangles formed from some number of these smaller squares, which ones can be obtained by a sequence of folds?