David Essner Exam 2   1982-1983

1. A die is rolled 4 times. If each of the numbers 1,2,3,4,5,6 is equally likely to come up on each toss, what is the probability the number 6 comes up exactly two times?

   (a) 1/18   (b) 25/216   (c) 1/3   (d) 25/324   (e) 1/9

2. Given $f(x) = 6x^3 - 5x^2 + x$, in which case is $f(x) > 0$ for both values of x?

   (a) 99/301 and 101/200   (b) 99/301 and 99/199   (c) 1/100 and 99/199
   (d) 151/300 and 149/300   (e) -1/100 and 301/99

3. A triangle has two sides of length 1 with 135° as the included angle. The length of the other side is

   (a) $(1 + \sqrt{2})^{1/2}$ (b) $(2 + \sqrt{2})^{1/2}$ (c) $\sqrt{2}$ (d) $\sqrt{5}$ (e) $(1 + \sqrt{3})^{1/2}$

4. The number of points in the intersection of the graphs of $x^2 - 4x + y^2 + 3 = 0$ and $x^2 - 2x + 4y^2 - 3 = 0$ is

   (a) 0   (b) 1   (c) 2   (d) 3   (e) more than 3

5. The system of equations: $x + 2y - 3z = 4$ has how many solutions for x,y,z?
   $$3x - y + 2z = 5$$
   $$x + 9y - 14z = 11$$

   (a) 0   (b) 1   (c) 2   (d) 3   (e) more than 3

6. An isosceles triangle has perimeter 10 and an angle of 120°. The length of the altitude to the side opposite the 120° angle is

   (a) $2\sqrt{3}$ (b) $10(2 - \sqrt{3})$ (c) $10 - 5\sqrt{3}$ (d) $10(\sqrt{3} - 1)$ (e) $\sqrt{3} + 1$

7. The sum of an arithmetic progression is 1275. If the first term is 3 and the last term is 99, then the difference between successive terms is

   (a) 2   (b) 3   (c) 4   (d) 5   (e) 6

8. The hypotenuse c and one leg a of a right triangle are consecutive integers; the area of the triangle is

   (a) $a\sqrt{c+a}/2$ (b) $(\sqrt{a^2+c^2} - 1)/2$ (c) $a\sqrt{c-1}/2$
   (d) $c\sqrt{a^2+1}/2$ (e) $\sqrt{(c-1)a}/2$

9. If the number 314_5 to the base 5 is expressed to the base 7 then the sum of its digits is
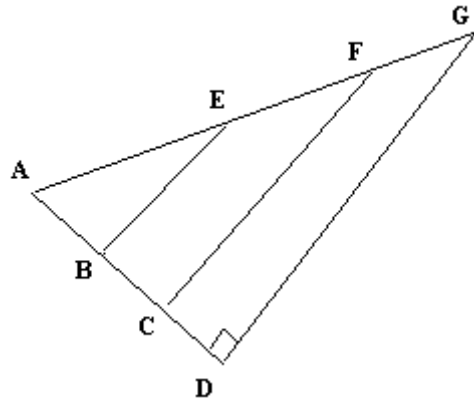
   (a) 5   (b) 6   (c) 7   (d) 8   (e) 9

10. If x is a very small positive number then $\dfrac{x+1/x-1}{x+2/x-3}$ is

   (a) very small   (b) very large   (c) near 3   (d) near 1/2   (e) near 1/3

11. In the pictured triangle $\mathbf{AE} = \mathbf{EF} = \mathbf{FG}$ and $\mathbf{AB} = \mathbf{BC} = \mathbf{CD}$. If $\mathbf{DG} = 5$ and $\angle \mathbf{ADG}$ is a right angle then $\mathbf{EB} + \mathbf{FC} =$

   (a) 40/9   (b) 5   (c) 17/3   (d) 6   (e) 31/6

12. A student took 6 exams. If the average on the first 3 exams was 82, and the average on the next two was 87, what score did the student need on the last exam to have an average of 85 for all six exams?

   (a) 84   (b) 86   (c) 87   (d) 88   (e) 90

13. Bill and John make 7 bets, doubling the amount of the bet each time. Bill wins the first, second, third and last bet while John wins the others. If Bill wins $3 then the amount bet the first time was

   (a) $3   (b) $2   (c) $1.50   (d) $0.50   (e) $0.20

14. Suppose a function f is defined by $f(1) = 2$, $f(2) = 3$, $f(3) = 5$ and $f(n) = f(n - 1) + f(n - 2)$ for $n \geq 4$. What is $f(8)$?
       (a) 112   (b) 55   (c) 15   (d) ≥ 500 (e) not enough information to tell

15. If p is a polynomial with real coefficients and no multiple roots, which of the following could be true?

       (a) p is of degree 6 and has exactly 3 real roots.
       (b) p is of degree 7 and has no real roots.
       (c) p is of degree 9 and has exactly 1 real root.
       (d) p is of degree 17 and has exactly 2 real roots.
       (e) p is of degree 6 and has exactly 7 real roots.

16. Consider the following three statements:

       A: The product of 3 consecutive integers is always divisible by 3.
       B: The product of 4 consecutive integers is always divisible by 4.
       C: The product of 5 consecutive integers is always divisible by 5.

Which of these are true?

       (a) All (b) None (c) A and B but not C (d) A and C but not B (e) only A

17. Given the parabola $y = x^2$, for which of the following values of a is it not possible to draw a straight line from the point $(2,a)$ tangent to the parabola?

       (a) 1 (b) -1 (c) 2 (d) 3 (e) 5

18. Suppose θ is an angle in the first quadrant for which $\tan \theta = 4$. What is $\cos 2\theta$?

       (a) 15/17 (b) -15/17 (c) $1/\sqrt{17}$ (d) 1/4 (e) -1/4
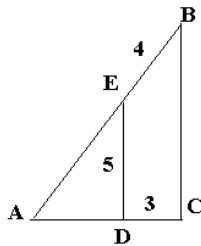
19. Of the following numbers, which is the largest?

       (a) 3^{4^5} (b) 4^{3^5} (c) 5^{4^3} (d) 5^{3^4} (e) 4^{5^3}

20. The negation of the converse of the statement "If $x < 0$ then $y < 0$" is equivalent to

       (a) $y < 0$ and $x \geq 0$ (b) $x > 0$ and $y \geq 0$ (c) $x < 0$ or $y < 0$
       (d) If $y \leq 0$ then $x > 0$ (e) $x < 0$ or $y \geq 0$

21. In the pictured triangle angles $\angle \mathbf{ADE}$ and $\angle \mathbf{ACB}$ are right angles, $\mathbf{BE} = 4$, $\mathbf{DC} = 3$, and $\mathbf{DE} = 5$. Then $\mathbf{AD} =$

       (a) 6   (b) $12/\sqrt{5}$ (c) $15/\sqrt{7}$ (d) 29/4   (e) $5\sqrt{2}$

22. In a class of 40 students every student takes at least one of the subjects Math, Science and English. Of these students 27 take Math, 15 take Math and Science, 19 take Math and English, and 17 either take only Math or else take all three. How many students take only Math?

(a) 5   (b) 6   (c) 7   (d) 8   (e) 9

23. Suppose a function f is defined by $f(1) = 1$, $f(2) = 2$, and $f(n) = 2f(n - 1) + f(n - 2)$ for $n \geq 3$. What is $f(100) - 2f(99) - 2f(97) - f(96)$?

(a) -1   (b) 0   (c) 1   (d) $\leq$ -100   (e) $\geq$ 100

24. A man owns a rectangular lot on a street. He wishes to fence the back and two sides (not the front). If he has 600 feet of fence, what is the largest area, in square feet, of lot he could fence in?

(a) 22,500   (b) 45,000   (c) 40,000   (d) 90,000   (e) 65,000

25. Denote by $[n]$ the number 111....1 (n ones expressing an integer number in decimal notation). Thus, for example, $[2] = 11$ and $[5] = 11,111$. Which one of the following is divisible by 7?
(a) [100]   (b) [82]   (c) [103]   (d) [14]   (e) [78]

26. If $p(x) = x^{2n} + ax^{2n-1} + ...$ is a polynomial of degree $2n$ which has as roots the numbers +1,-1,+2,-2,+3,-3,...,+n,-n then $a =$

(a) $n(n + 1)/2$ (b) 0   (c) 1   (d) -1   (e) n^2

27. What is the value (sum) of the infinite series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n(n + 2)}$?

(a) 1   (b) 1/2   (c) ∞ (d) $\pi^2/6$ (e) 3/4

28. Suppose p is a polynomial of degree 6 with real coefficients such that $p(-1)= 3$, $p(0) = -2$, $p(4) = 1$, $p(6) = -2$. Let n be the number (counting multiplicities) of real roots of p. Which of the following is the most we can say about n?

(a) $n \leq 6$ (b) $n \geq 1$ (c) $3 \leq n \leq 6$ (d) $4 \leq n \leq 6$ (e) $2 \leq n \leq 4$

29. In the Cartesian plane the two tangent lines from the point (2,2) to the circle $x^2 + y^2 = 1$ intersect the circle at the points (a,b) and (c,d). Then $a + b =$

(a) 1/2   (b) 1/4   (c) 2/3   (d) 1/3   (e) 0

30. The number $(1.001)^{100}$ is best approximated by

(a) 1.010   (b) 1.101   (c) 1.111   (d) 2.0   (e) 1.0011?

31. If the value of an investment at an annual rate of r% compounded semi-annually is to double after one year, then r is nearest the rate

(a) 67   (b) 75   (c) 83   (d) 91   (e) 50

32. What is the value of the infinite product $\prod_{n=3}^{\infty}(1+\dfrac{1}{n(n-2)})$?

(a) 2   (b) ∞ (c) 0   (d) 1   (e) 4

33. For which of the listed integers n is it <u>not</u> possible to find integers x and y with $x^2 - y^2 = n$?

(a) 22   (b) 21   (c) -21   (d) 7   (e) 103

34. Suppose a and b are integers such that the polynomial
$$x^6 + ax^5 + bx^4 + ax^3 - bx^2 - 1$$
has a positive integral root. Then $a =$

(a) 1   (b) -1   (c) 0   (d) 2   (e) there are several values depending on b

35. A poker hand consists of 5 cards from a standard deck of 52 cards. Let
    $M =$ number of poker hands with exactly one pair (no triples, four of a kind, or other pairs)
    $N =$ number of poker hands with exactly two pairs (no triples)
    $P =$ number of poker hands with three of a kind (no four of a kind or other pairs)
Then $(M - N - P)/(13 \times 44 \times 8) =$

(a) -100   (b) 1362   (c) 201   (d) 250   (e) 512