David Essner Mathematics Competition XXXII, 2012-2013

Preliminary Exam

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1. How many ways are there of choosing a, b from $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ such that $a < b$ and $a + b$ is a multiple of 3?

(a) 6 (b) 9 (c) 12 (d) 18 (e) 36

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2. John has taken a total of n tests each worth 100 points, and his average was 78. If on the last test he had made 91 instead of 70, then his average would have been 81. What is n?

(a) 7 (b) 6 (c) 5 (d) 4 (e) 3

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3. Pat sold two DVDs on eBay for \$24 each. If she made a profit of 20% on one and a loss of 20% on the other, then what was the net result of the sale of the two items?

(a) \$1 profit (b) \$1 loss (c) \$2 profit (d) \$2 loss (e) no profit or loss

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4. If $\dfrac{x}{y-6} = \dfrac{y}{z-3} = \dfrac{z}{x-4} = 2$ then x equals

(a) 12 (b) 11 (c) 10 (d) 9 (e) 8

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5. Jar A has 2 gallons and Jar B has 3 gallons of acid solution. If the fraction of acid in Jar A is 2/5 and the fraction of acid in the combined contents of Jar A and Jar B would be 1/2, then what is the fraction of acid in Jar B?

(a) 8/15 (b) 3/5 (c) 7/10 (d) 14/25 (e) 17/30

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6. For how many integers x from 1 to 100 is $x^2 + x$ a multiple of 7?

(a) 21 (b) 25 (c) 28 (d) 32 (e) 36

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7. In how many ways can 8 persons be grouped into 4 pairs?

(a) 128 (b) 105 (c) 96 (d) 72 (e) 64

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8. Given a triangle ΔABC, if sides AB and AC each have length 3 and BC has length 2 then the length of the altitude from B to side AC is

(a) 3/2 (b) $\sqrt{3}$ (c) $4\sqrt{2}/3$ (d) $2\sqrt{3}/3$ (e) 7/4

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9. If a, b are positive integers, $a \leq b \leq 9$, then how many different triangles are there with sides of length a, b, and 9?

(a) 25 (b) 22 (c) 32 (d) 29 (e) 45

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10. The graphs of $y = x - 3$ and $y = mx + 1$ meet at a point whose x and y coordinates are each positive if and only if m is in the interval

(a) $(-1, 3)$ (b) $(1/3, 1)$ (c) $(1/3, 3)$ (d) $(-1/3, 1)$ (e) $(-1, 1/3)$

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11. If $|x - 1| < .1$ and $x \neq 1$ then $(x^2 + 3x - 4)/(x^2 - x)$ must satisfy

(a) $x < 0$ (b) $0 \leq x < 2$ (c) $2 \leq x < 4$ (d) $4 \leq x < 6$ (e) $6 \leq x$

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12. Two dice are rolled; the value on each die is a random integer 1 through 6. If x is the value on one die and y is the value on the other die, then what is the probability that $|x - y|$ is less than 2?

(a) 1/3 (b) 1/2 (c) 7/18 (d) 5/12 (e) 4/9

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13. Which number is the largest?

(a) 2^{48} (b) 3^{42} (c) 4^{30} (d) 6^{24} (e) 9^{18}

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14. What is the largest prime factor of $2^{16} - 2^{13} + 2^8$?

(a) 5 (b) 7 (c) 11 (d) 13 (e) 17

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15. If a, b are positive integers and $(a + 2b)(a - b) = 27$, then $2a + b$ is:

(a) 6 (b) 9 (c) 12 (d) 15 (e) not uniquely determined

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16. If ΔABC is a right triangle with side AB having length 3 and hypotenuse AC having length 5, then what is the length of the angle bisector AD of $\angle BAC$, where D is on the side BC?

(a) 4 (b) $\sqrt{13}$ (c) $4\sqrt{5}/3$ (d) $3\sqrt{5}/2$ (e) $5\sqrt{3}/2$

17. A box contains 4 balls, one each of the colors light blue, dark blue, light red, and dark red. A ball is drawn at random, returned to the box, and then a second ball is drawn at random. If at least one of the drawn balls was light blue, what is the probability that neither of the drawn balls was a shade of red?

(a) 3/8 (b) 3/7 (c) 2/7 (d) 1/3 (e) 1/4

18. If $12x = 0.12121212...$ (the digits 12 repeat indefinitely) then x equals

(a) 1/99 (b) 1/101 (c) 0.0099 (d) 0.011 (e) 0.012

19. Let N be the smallest positive integer such that if N is divided by 7 the remainder is 3 and if N is divided by 11 the remainder is 4. The sum of the digits of N is

(a) 7 (b) 14 (c) 11 (d) 9 (e) 12

20. If the polynomial $x^2 - 4x + 2$ has roots r and s and the polynomial $x^2 + bx + c$ has roots $r + 1$ and $s + 1$, then $b + c$ equals

(a) 1 (b) 2 (c) 3 (d) 5 (e) 7

21. In the binomial expansion of $(x^2 - y^3)^8$ what is the coefficient of the term $x^a y^b$ with $a - b = 1$?

(a) 28 (b) -28 (c) 56 (d) -56 (e) 0

22. In the Cartesian plane, if a line through the point $(10, 0)$ is tangent to the circle $x^2 + y^2 = 25$ at the point (a, b) then a is

(a) 5/2 (b) $5/\sqrt{2}$ (c) $5\sqrt{2}/4$ (d) $5\sqrt{3}/2$ (e) not uniquely determined

23. In the Cartesian plane, what is the radius of the circle that passes through the points $(0, 0)$, $(4, 0)$, and $(0, -2)$?

(a) 3 (b) 7/3 (c) 5/2 (d) $\sqrt{7}$ (e) $\sqrt{5}$

24. If m, n are positive integers and $n! = 210m!$ then $m + n$ equals

(a) 8 (b) 11 (c) 12 (d) 15 (e) 21

25. For any real number x, let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x. If $a^2\lfloor a \rfloor = 25/2$ and $b\lfloor b \rfloor = 10$ then $b - a$ equals

(a) 7/2 (b) 11/4 (c) 11/2 (d) 7/8 (e) 5/6

26. The two equations $|y - x| = 1$ and $y/x = xy$ have how many simultaneous solution pairs (x, y)?

(a) 6 (b) 4 (c) 3 (d) 2 (e) 1

27. A sequence of 9 terms has the property that, except for the first two, each term is the sum of all the preceding terms in the sequence. If the 9th term is 512, then the sum of the first two terms is

(a) 4 (b) 6 (c) 8 (d) 12 (e) 16

28. Let $f(x)$ denote a real valued function. If $f(1 + f(x)) = 2f(x)$ and $f(0) = -2$ then $f(-3)$ equals

(a) -8 (b) -6 (c) -4 (d) -1 (e) 3

29. A circle of radius 1 and a square have the same center and equal areas. If the circle intersects one side of the square in points P and Q, then what is the length of the segment PQ?

(a) $\pi/2$ (b) $\pi - 3/2$ (c) $(\pi - 3)/2$ (d) $\sqrt{4 - \pi}$ (e) $\frac{1}{2}\sqrt{1 + \pi}$

30. If $\log_2 x^4 + \log_x 2 = 4$ then x equals

(a) 1/4 (b) $\sqrt{2}$ (c) 2 (d) 1/2 (e) $2\sqrt{2}$