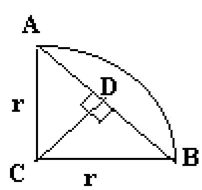
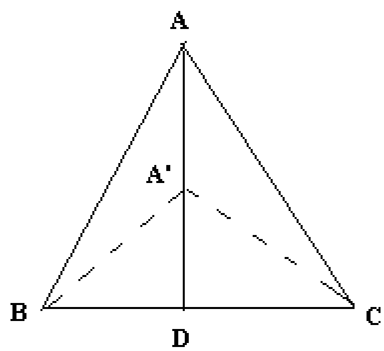
Answers and Brief Solutions to E1982

1. (d) $\log_a a^{10} = \log_a x^2$ gives $10 = 2 \log_a x$
2. (c) By similar triangles $|AC|/|AE| = |BC|/|DE| = 3/4$; hence $|AC| = 7\ 1/2$
3. (b) Four times each year the interest is (12/4)% = 3% of the value, and each time the new value is the old value multiplied by 1.03.
4. (d) The sum = 10 + 12 + … + 100 = [(10 + 100)/2]x46. Note (10 + 100)/2 is the average term and 46 is the number of terms
5. (b) $n!/(n-2)! = [1 \times 2 \times \ldots \times (n-2) \times (n-1) \times n]/[1 \times 2 \times \ldots \times (n-2)] = (n-1) \times n$.
6. (d) The total score for the first 3 exams is 3x90 = 270 and for all 5 exams is 5x88 = 440 leaving 170 for the last two exams.
7. (c) $2^{10} = 1024 – 10^3$ so $2^{40} = (2^{10})^4 \approx (10^3)^4 = 10^{12}$
8. (b) $(x-1)^2 + y^2 = x^2 + (y-2)^2$ by the distance formula
9. (d) $f(1) = 1 > 0 > -1 = f(2)$ so there is a root between 1 and 2
10. (e) $\sin\theta = 3/5$, $\cos\theta = 4/5$ and $\tan 2\theta = \sin 2\theta/\cos 2\theta = (2 \sin\theta \cos\theta)/(\cos^2\theta - \sin^2\theta)$
11. (d) If d is the distance between A and B, and x is the unknown speed then
    $2d/35 = d/30 + d/x$ is the total time. Solve for x.
12. (d) $e^{\pi i} = \cos\pi + i \sin\pi$
13. (a) $f(3) = f(2) - 2f(1) + f(0) = 4 - 6 + 2 = 0$; $f(4) = f(3) - 2f(2) + f(1) = 0 - 8 + 3 = -5$; $f(5) = f(4) - 2f(3) + f(2) = -5 + 4 = -1$
14. (b) All powers of x must be even or all must be odd.
15. (b) 3*2 = 3x2 – 3 – 2 = 1; (3*2)*3 = (3*2)x3 – (3*2) – 3
16. (e) $x + 2 = 3(x-2)$ or $x + 2 = -3(x-2)$
17. (e) Amounts of alcohol A and water W at successive stages are $12W$, $3A \to 12W$, $8A \to 9W$, $6A \to 9W$, $11A \to 27/4\ W$, $33A$.
18. (b) (See figure below) $|AD| = |BD| = 1$ and the area of the triangle ABC $= 2(1/2)(1)|CD|$ gives $|CD| = 1$. Hence $r = \sqrt{2}$ and $\angle ACB = 90°$ gives the area of the sector as $(90/360)(\pi)(\sqrt{2})^2$



19. (d) After n payments Ed has $1 + 2 + \ldots + 2^{n-1} = 2^n - 1$ and Dona has $22 + 15n$.
    If $n \geq 7$ then $2^n - 1 \geq 22 + 15n$
20. (d) Divide the numerator and denominator of the fraction by x^2. For x large $1 + 3/x + 12/x^2$ is near 1 and $2 + 5/x + 3/x^2$ is near 2, so the fraction is near 1/2.
21. (e) Let x = .3212121… . Then 100 - x = 32.1212… - .3212… = 31.8 so x = 31.8/99 = 318/990 = 53/165. Another method is $x = 3/10 + 21/1000\ (1 + 1/100 + 1/(100)^2 + \ldots) = 3/10 + (21/1000)(100/99)$
22. (b) $a_1 - b_1, a_2 - b_2, \ldots$ is also an arithmetic progression, so the sum is 3 + 4 + 5 + 6 + 7 + 8 + 9 = 42

23. (See the figure) Triangles ABC and $A'BC$ both satisfy the conditions of the problem, but have different areas.



24. (d) If the roots of the polynomial are r_1, r_2, r_3 then the polynomial is $(x - r_1)(x - r_2)(x - r_3)$, the constant term is $- r_1 r_2 r_3$ and the coefficient of x^2 is $(r_1 + r_2 + r_3)$

25. (a) The expansion is the sum of terms $C(5,n)(x^2)^n(-2/x^3)^5$ where $C(5,n) = 5!/(n!(5 - n)!)$. For the constant term, $2n = 3(5 - n)$ so $n = 3$ and the coefficient is $C(5,3)(-2)^2$

26. (c) If s is the side of the square then the radius of C is $s/2$ and the side of E is $(2)(s/2)(\sqrt{3}/2) = s\sqrt{3}/2$. The area of E is then $(\sqrt{3}/2)(s\sqrt{3}/2)(s\sqrt{3}/4) = 3\sqrt{3}\,s^2/16$

27. (d) The inequality is equivalent to $a^3b + ab^3 - a^4 - b^4 \geq 0$. Factoring gives $-(a^3 - b^3)(a - b) \leq 0$. If $a < b$ or $a > b$ then $(a^3 - b^3)(a - b) > 0$.

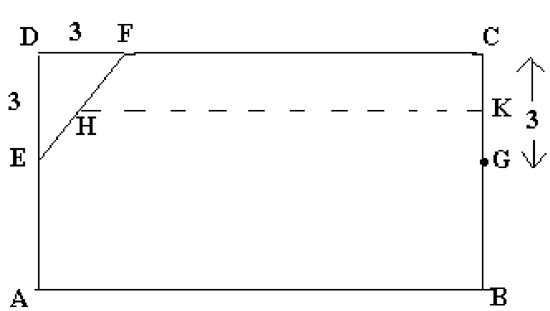
28. (d) This may be easily seen by sketching the ellipse and parabola and noting the four points of intersection.

29. (e) $(1 + a)^b - 1 + ba$ if a is small, by the binomial theorem.

30. (c) The exponential 2^x grows more rapidly than powers of x or logarithms.

31. (b) If the point is (a,b)then $b = 2a$ and the line from (4,2) to (a,b) should be perpendicular to $y = 2x$, so $(2a - 2)/(a - 4) = -1/2$.

32. (d) See the figure. $|HK| = 5 + (1/2 \times 3)$; $|CK| = 1/2 \times 3 = |KG|$ and therefore $|HG| = (169/4 + 9/4)^{1/2}$.



33. (e) If x is the given number then $x^2 = (5 + 2\sqrt{6}) - 2[(5 + 2\sqrt{6})(5 - 2\sqrt{6}]^{1/2} + (5 - 2\sqrt{6})$
34. (d) If a divides both $n - 11$ and $n + 49$ then a divides $(n + 49) - (n - 11) = 60$
35. (c) $a_3 = 7$, $a_4 = 15$, $a_5 = 31$,…. In general $a_n = 2^n - 1$ so $(a_n)^{1/n} = (2^n - 1)^{1/n} \approx (2^n)^{1/n} = 2$ for n large.
36. (a) If $p(x)$ is the polynomial then $p(x) = p(-x)$ for all numbers x, so if $p(x)$ has a minimum at $x = r$ then it has the same minimum at $x = -r$.
37. (b) $12 = y^2 - 2xy = y(y - 2x)$ so y divides 1 and is 1,2,3,4,6 or 12. Of these only $y = 6$ makes x a positive integer.
38. (e) There are 13 denominations, so there are $13 \times ((4 \times 3)/2) = 78$ possible pairs. The remaining 3 cards can be chosen in $(48 \times 44 \times 40)/6$ ways so as to have no more pairs, giving $78 \times 8 \times 44 \times 40$ hands
39. (b) Only f_1 is additive
40. (b) $Q_n = Q_0 r^n$, where Q_0 is the initial amount, Q_n the amount after n years, and r the growth rate. Since $Q_8 = 4Q_0 = Q_0 r^8$, $r = 4^{1/8} = 2^{1/4}$. Solve $8\sqrt{2} = 2^{n/4}$ for n.