Answers and Brief Solutions to E1983

1. (b) \((1/6)^2 (5/6)^2 = 25/216\)

2. (a) \(f(x) = x(3x - 1)/(2x - 1)\). To make \(f(x) > 0\) a given \(x\) must lie between 0 and 1/3 or be > 1/2

3. (b) Use the identity \(\cos 2A = 2 \cos^2 A - 1\) with \(A = 22 1/2°\). Then \(\sqrt{2}/2 = \cos 45° = 2 \cos^2 22 1/2° - 1 = 2(x/2)^2 - 1\) where \(x\) is the length of the other side.

4. (d) This is the intersection of the circle \((x - 2)^2 + y^2 = 1\) and the ellipse \((x - 1)^2/4 + y^2 = 1\)

5. (e) Add –3 times the first equation to the second and –1 times the first equation to the third. Then add the second and third equations. An equivalent system is therefore \(x + 2y - 3z = 4\) and \(-7y + 11z = -7\) which has infinitely many solutions (i.e. assign any value to \(z\) and solve for \(x\) and \(y\)).

6. (c) \(\cos 30° = (5 - 2x)/2x = \sqrt{3}/2\) gives \(x = 10 - 5\sqrt{3}\)

7. (c) Let \(r\) be the difference. Then \(1275 = [(99+3)/2] \times (96/r) = \text{product of the average term and the number of terms.} \) Solve for \(r\).

8. (a) If \(A\) is the area then \(A = 1/2 a(c^2 - a^2)^{1/2} = 1/2 a [(a + 1)^2 - a^2]^{1/2} = 1/2 a(2a + 1)^{1/2}\)

9. (b) \(3\times 5^2 + 1\times 5 + 4 = 84 = 1\times 7^2 + 5\times 7 + 0\times 7^0\)

10. (d) If \(x\) is small the fraction is approximated by \((1/x)/(2/x)\)

11. (b) Using properties of similar triangles, \(EB + FC = (1/3) \times 5 + (2/3) \times 5 = 5\)

12. (e) The student scored 3(87 – 85) = 9 points below an 85 average on the first three exams and 2(87 – 85) = 4 above an 85 average on the next two. The net is 5 points below, so 85 + 5 = 90 is needed on the last exam.

13. (e) Let \(x\) be the first bet. Then Bill wins \(x + 2x + 4x + 64x = 71x\) and John wins \(8x + 16x + 32x = 56x\). The net win is \(71x - 56x = 15x = 3\) for Bill. Solving gives \(x = 0.20\).

14. (a) By successive computations \(f(4) = 2 + 3 + 5 = 10, f(5) = 18, f(6) = 33, f(7) = 61\) and \(f(8) = 112\).

15. (c) The number of complex roots = \(p - \text{number of real roots}\) must be an even number.

Also the total number of roots is \(\leq p\).

16. (a) \(A\) is true since for any three consecutive integers, one is divisible by 3. Similarly for \(B\) and \(C\).

17. (e) Since \(5 > 2^2\), the point (2,5) lies inside the parabola.

18. (b) \(\cos 2\theta = 2\cos^2 \theta - 1\). From \(\tan \theta = 4\), \(\cos \theta = 1/\sqrt{17}\)

19. (a) \(3^5 > 5^3\) and \(3^4 > 4^3\) eliminates (e) and (c). From \(\frac{\log 3^{(4^3)}}{\log 4^{(3^5)}} = (4/3)^5 (\log 3/\log 4)\) = \((4/3)^5 \log_4 3\) it is seen that \(3^{(4^3)} > 4^{(3^5)}\). Similarly \(3^{(4^5)} > 5^{(3^4)}\)

20. The converse is “If \(y < 0\) then \(x < 0\)” and this is equivalent to “\(y > 0\) or \(x < 0\)”. The negation is equivalent to “\(y < 0\) and \(x \geq 0\)”

21. (c) Let \(AD = x\). Then \(AE = (25 + x^2)^{1/2}\) and by similar triangles \((x + 3)/[(25 + x^2)^{1/2} + 4] = x/(25 + x^2)^{1/2}\). Solve for \(x\).
22. (a) In the diagram below, \( a + b + d + g = 27 \), \( b + d = 15 \), \( d + g = 19 \), and \( a + d = 17 \). Solving gives \( a = 5 \).

23. (b) The given expression equals \( [f(100) - 2f(99) - f(98)] + [f(98) - 2f(97) - f(96)] = 0 + 0 = 0 \)

24. (b) Let \( A \) be the area. Then \( A = xy \) and \( 2x + y = 600 \). Thus \( A = x(600 - 2x) \) or \( A = 45,000 - 2(x - 150)^2 \). The maximum value is 45,000 since \(-2(x - 150)^2\) is non-positive.

25. (e) By long division \([6]/7 = 111111/7\) has a zero remainder, and 6 is the smallest positive integer for which this is true. Therefore 7 divides \( n \) if and only if 6 divides \( n \).

26. (b) \( p(x) = (x - r_1)(x - r_2) \ldots (x - r_{2n}) \) where the \( r_j \) are the roots of \( p \). The coefficient of \( x^{2n-1} \) is \( a = -(r_1 + r_2 + \ldots + r_n) \) so \( a = 0 \).

27. (e) \( 1/n(n+2) = 1/2 (1/n - 1/(n+2)) \). Therefore the sum is \( 1/2 \left[(1 - 1/3) + (1/2 - 1/4) + (1/3 - 1/5) + (1/4 - 1/6) + \ldots\right] = 1/2 \ (1 + 1/2) = 3/4 \). (All other terms cancel)

28. (d) In general if \( p(a) < 0 < p(b) \) then \( p \) must have a root between \( a \) and \( b \). Therefore there are roots between \(-1 \) and \( 0 \), between \( 0 \) and \( 4 \), and between \( 4 \) and \( 6 \). There must be at least one more real root since \( n = 6 \) and the number of real roots is even. (complex roots come in pairs). The total number of different roots is \( \leq \) the degree of \( p \).

29. (a) At the point \((a,b)\) the slope of the tangent line is \(-a/b\). This must equal the slope of the line from \((2,2)\) to \((a,b)\) so \(-a/b = (2-b)/(2-a)\). Solving gives \( 2a + 2b = a^2 + b^2 = 1 \) (since \(a,b\) is on the circle).

30. (b) If \( a \) is small and \( b \) not too large then \((1+a)^b \approx 1 + ba\) (by the binomial theorem).

31. (c) If \( r \) is the rate then \((1+r/2)^2 = 2\) gives \( r = 2(\sqrt{2} - 1) \approx 0.828 \)

32. (a) \( 1 + 1/n(n-2) = (n^2 - 2n + 1)/n(n-2) = (n-1)^2/n(n-2) \). Thus the product is \((2^2)/(1\times3) \times (3^2)/(2\times4) \times (4^2)/(3\times5) \times (5^2)/(4\times6) \times (6^2)/(5\times7) = 2\)

(all other terms cancel)

33. (a) If one of \(x,y\) is odd and the other even then \(x^2 - y^2\) is odd. If both \(x, y\) are odd then \(x^2 - y^2\) has the form \((2m + 1)^2 - (2n + 1)^2 = 4(m^2 + m - n^2 - n)\) and is divisible by 4. Similarly if both \(x, y\) are even then \(x^2 - y^2\) is divisible by 4. Therefore \(x^2 - y^2\) is never divisible by two consecutive squares.

22. From the identity \((n + 1)^2 - n^2 = 2n + 1\) it is clear that every odd integer is the difference of two consecutive squares.

34. (c) From \(x^2(x^4 + ax^3 + bx^2 + ax - b) = 1\) if \(x\) is an integer root then \(x^2\) divides 1. Therefore \(x = 1\) is a root and \(1 + a + b + a - b - 1 = 0\) gives \(a = 0\).

35. (c) \(M = 13\times6\times(48\times44\times40/6)\), \(N = (13\times12/2)\times6\times6\times44\) and \(P = 13\times4\times(48\times44/2)\)