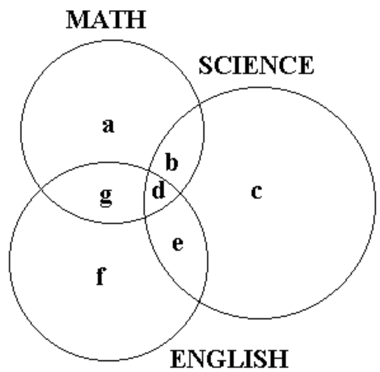
Answers and Brief Solutions to E1983

1. (b) $(1/6)^2(5/6)^2 = 25/216$
2. (a) $f(x) = x(3x - 1)/(2x - 1)$. To make $f(x) > 0$ a given x must lie between 0 and 1/3 or be > 1/2
3. (b) Use the identity $\cos 2A = 2\cos^2 A - 1$ with $A = 22\ 1/2\ °$. Then $\sqrt{2}/2 = \cos 45° =$
   $2\cos^2 22\ 1/2\ ° - 1 = 2(x/2)^2 - 1$ where x is the length of the other side.
4. (d) This is the intersection of the circle $(x - 2)^2 + y^2 = 1$ and the ellipse
   $(x - 1)^2/4 + y^2 = 1$
5. (e) Add –3 times the first equation to the second and –1 times the first equation to the third. Then add the second and third equations. An equivalent system is therefore $x + 2y - 3z = 4$ and $-7y + 11z = -7$ which has infinitely many solutions (i.e. assign any value to z and solve for x and y).
6. (c) $\cos 30° = (5 - 2x)/2x = \sqrt{3}/2$ gives $x = 10 - 5\sqrt{3}$
7. (c) Let r be the difference. Then $1275 = [(99+3)/2] \times (96/r)$ = product of the average term and the number of terms.. Solve for r.
8. (a) If A is the area then $A = 1/2\ a(c^2 - a^2)^{1/2} = 1/2\ a\ [(a + 1)^2 - a^2\ ]^{1/2} = 1/2\ a(2a + 1)^{1/2}$
9. (b) $3\times5^2 + 1\times5 + 4 = 84 = 1\times7^2 + 5\times7 + 0\times7^0$
10. (d) If x is small the fraction is approximated by $(1/x)/(2/x)$
11. (b) Using properties of similar triangles, $EB + FC = (1/3)\times5 + (2/3)\times5 = 5$
12. (e) The student scored $3(87 - 85) = 9$ points below an 85 average on the first three exams and $2(87 - 85) = 4$ above an 85 average on the next two. The net is 5 points below, so $85 + 5 = 90$ is needed on the last exam.
13. (e) Let $\$x$ be the first bet . Then Bill wins $x + 2x + 4x + 64x = 71x$ and John wins $8x + 16x + 32x = 56x$. The net win is $71x - 56x = 15x = \$3$ for Bill. Solving gives $x = \$0.20$.
14. (a) By successive computations $f(4) = 2 + 3 + 5 = 10$, $f(5) = 18$, $f(6) = 33$, $f(7) = 61$ and $f(8) = 112$.
15. (c) The number of complex roots = p – number of real roots must be an even number. Also the total number of roots is $\le p$.
16. (a) A is true since for any three consecutive integers, one is divisible by 3. Similarly for B and C.
17. (e) Since $5 > 2^2$, the point (2,5) lies inside the parabola.
18. (b) $\cos 2\theta = 2\cos^2\theta - 1$. From $\tan\theta = 4$, $\cos\theta = 1/\sqrt{17}$
19. (a) $3^5 > 5^3$ and $3^4 > 4^3$ eliminates (e) and (c). From $\dfrac{\log 3^{(4^5)}}{\log 4^{(3^5)}} = (4/3)^5 (\log 3/\log 4) =$
   $(4/3)^5 \log_4 3$ it is seen that $3^{(4^5)} > 4^{(3^5)}$ Similarly $3^{(4^5)} > 5^{(3^4)}$
20. The converse is “If $y < 0$ then $x < 0$” and this is equivalent to “$y > 0$ or $x < 0$”. The negation is equivalent to “$y < 0$ and $x \ge 0$”
21. (c) Let $AD = x$. Then $AE = (25 + x^2)^{1/2}$ and by similar triangles
   $(x + 3)/[(25 + x^2)^{1/2} + 4] = x/(25 + x^2)^{1/2}$. Solve for x.

22. (a) In the diagram below, $a + b + d + g = 27$, $b + d = 15$, $d + g = 19$, and $a + d = 17$. Solving gives $a = 5$.



23. (b) The given expression equals $[f(100) - 2f(99) - f(98)] + [f(98) - 2f(97) - f(96)] =$
$0 + 0 = 0$

24 . (b) Let A be the area. Then $A = xy$ and $2x + y = 600$. Thus $A = x(600 - 2x)$ or $A = 45{,}000 - 2(x - 150)^2$. The maximum value is 45,000 since $-2(x - 150)^2$ is non-positive.

25. (e) By long division [6]/7 = 111111/7 has a zero remainder, and 6 is the smallest positive integer for which this is true. Therefore 7 divides $[n]$ if and only if 6 divides n.

26. (b) $p(x) = (x - r_1)(x - r_2)\ldots.(x - r_{2n})$ where the r_j are the roots of p. The coefficient of x^{2n-1} is $a = -(r_1 + r_2 + \ldots + r_n)$ so $a = 0$.

27. (e) $1/n(n + 2) = 1/2\ (1/n - 1/(n + 2))$. Therefore the sum is $1/2\ [(1 - 1/3) + (1/2 - 1/4) + (1/3 - 1/5) + (1/4 - 1/6) + \ldots] = 1/2\ (1 + 1/2) = 3/4$. (All other terms cancel)

28. (d) In general if $p(a) < 0 < p(b)$ then p must have a root between a and b. Therefore there are roots between –1 and 0 , between 0 and 4, and between 4 and 6. There must be at least one more real root since $n = 6$ and the number of real roots is even. (complex roots come in pairs). The total number of different roots is ≤ the degree of p.

29. (a) At the point (a,b) the slope of the tangent line is $-a/b$. This must equal the slope of the line from (2,2) to (a,b) so $-a/b = (2 - b)/(2 - a)$. Solving gives $2a + 2b = a^2 + b^2 = 1$ (since (a,b) is on the circle).

30. (b) If a is small and b not too large then $(1 + a)^b \approx 1 + ba$ (by the binomial theorem).

31. (c) If r is the rate then $(1 + r/2)^2 = 2$ gives $r = 2(\sqrt{2} - 1) \approx 0.828$

32. (a) $1 + 1/n(n - 2) = (n^2 - 2n + 1)/n(n - 2) = (n - 1)^2/n(n - 2)$. Thus the product is
$(2^2)/(1\times3) \times (3^2)/(2\times4) \times (4^2)/(3\times5) \times (5^2)/(4\times6) \times (6^2)/(5\times7) = 2$
(all other terms cancel)

33. (a) If one of x,y is odd and the other even then $x^2 - y^2$ is odd. If both x and y are odd then $x^2 - y^2$ has the form $(2m + 1)^2 - (2n + 1)^2 = 4(m^2 + m - n^2 - n)$ and is divisible by 4. Similarly if both x and y are even then $x^2 - y^2$ is divisible by 4. Therefore $x^2 - y^2$ is never 22. From the identity $(n + 1)^2 - n^2 = 2n + 1$ it is clear that every odd integer is the difference of two consecutive squares.

34. (c) From $x^2\times(x^4 + ax^3 + bx^2 + ax - b) = 1$ if x is an integer root then x^2 divides 1. Therefore $x = 1$ is a root and $1 + a + b + a - b - 1 = 0$ gives $a = 0$.

35. (c) $M = 13\times6\times(48\times44\times40/6)$, $N = (13\times12/2)\times6\times6\times44$ and $P = 13\times4\times(48\times44/2)$