Answers and Brief Solutions to E1984

1. (e) The probability that the second dice differs from the first is 5/6, and then the third from the first two is 4/6. The answer is (5/6)(4/6) = 5/9
2. (d) The fraction is approximated by x/x since the other terms in the numerator and denominator are relatively small.
3. (c) A number is divisible by $4 = 2^2$ and $6 = 2\times3 = 12$ provided it is divisible by $2^2\times3 = 12$. Division of 100 by 12 gives 8, with remainder.
4. (c) The ratio r satisfies $r^2a = b$. The first term is then $a/r^3 = \dfrac{a}{(b/a)^{3/2}}$.
5. (a) The graphs of the line $y = mx + 1$ and the ellipse $4x^2 + y^2 - 4 = 0$ intersect for all numbers m.
6. (b) Using the binomial approximation $(1 + x)^n \approx 1 + nx + \dfrac{n(n-1)}{2}x^2$ the second terms all have magnitude 0.1 . The third terms has smallest magnitude for $(1.001)^{100}$.
7. (d) A quick mental solution is $80 + \dfrac{6(84-80)}{2}$
8. (c) If x is the side then there is a right triangle with hypotenuse x and legs $x - 1$ and $x/2$. Using the Pythagorean Theorem $x = 4$.
9. (a) $M - N = 1000(d - a) + 100(c - b) + 10(b - c) + (a - d)$. Since the first three terms are even, the last term must also be even.
10. (a) Since the triangles are similar the lower triangle has larger area if its hypotenuse is > 6. Thus $\cos\theta < 3/6$.
11. (d) The answer follows from $(1 + r)^n = 2$ (the rate r is interpreted for each interest period). If the rate were annual, the usual interpretation, then $(1 + r/n)^n = 2$ gives $r = n(2^{1/n} - 1)$.
12. (e) This makes use of the properties (*1*) "if P then Q" is equivalent to "if not Q then not P" and (*2*) "not P and not Q" is equivalent to "not (P or Q)"
13. (b) $= 13\times12\times C(4,3)C(4,2)$ where 13×12 is the number of ways to choose the two ranks and $C(m,n)$ is the number of combinations of n objects from m.
14. (a) If θ is the angle then $\sin(\theta/2) = \sqrt{5}/3$, $\cos(\theta/2) = 2/3$ gives $\sin\theta = 2(\sqrt{5}/3)(2/3) = 4\sqrt{5}/9$ and $\cos\theta = 4/9 - 5/9 = -1/9$. Thus $\tan\theta = \sin\theta/\cos\theta = -4\sqrt{5}$.
15. (c) $x = -2i$ is also a root and $(x - 2)(x - 2i)(x + 2i) = x^3 - 2x^2 + 4x - 8$ which when divided into the given polynomial gives $x + 3$.
16. (c) $1553 = 1024 + 512 + 16 + 1 = 2^{10} + 2^9 + 2^4 + 2^0$
17. (d) Let n be the number. Then, for some positive integer p, $n/5 + np/7 + 12 = n$, or $n(28 - 5p) = 420$. This can be solved if $p = 5$.
18. (b) There are $9^3 = 729$ three digit numbers which have no 1's, but this includes 000. Thus the answer is 999 – (729 – 1).
19. (a) The line between the centers bisects the chord and is formed of two parts. One part is a leg of length 6 computed from a right triangle with hypotenuse 10 and one leg 8; the other is a leg of length 15 computed from a right triangle with hypotenuse 17 and one leg 8.
20. (b) $= 3^{2x+9} - 3^{2x+7} = 3^{2x+7}(3^2 - 1)$

21. (c) After adding, the amount of acid is 25 and of solution is 50 + x. Thus $\frac{25}{50+x}$ = 0.4 .

22. (c) Divide by 5 to get $x + 7y/5 = 93$. If y is a positive integer multiple of 5, $1 \le y \le 13$, then there is an integer solution for x.

23. (e) Tom pays $1.05x$ and Bill pays $(.9)(1.05)x$. Thus $(1.05)(.1)x = .70$.

24. (b) $f(x) = (1 + x)^5 = (1 + 1/3)^5 = (4/3)^5$.

25. (d) Let r be the base. Then $r^3 - (4r^2 + 4r) = 3r^2 + 4r$ gives $r^2 - 7r - 8 = 0$; hence $r = 8$.

26. (d) Let $x = \log_5 12$. Then $5^x = 12$ and $x \log_{10}5 = \log_{10}12$. It follows that $x = \frac{2\log_{10}2+\log_{10}3}{\log_{10}10-\log_{10}2} = \frac{2a+b}{1-a}$.

27. (b) The region consists of the interior of the triangle with vertices (0,0), (2,4), and (3,3). Only (2,3) is in this set.

28. (a) Each side of the inscribed rectangle is $\sqrt{2}$ and of the circumscribed rectangle is 2. The average area is $1/2\ (2^2 + \sqrt{2}^{\,2}) = 3$.

29. (a) Each win multiplies the amount by 3/2 and each loss by 1/2. Thus Tom ends with $64(3/2)^2(1/2)^3 = 18$.

30. (b) Let r, t be the rate and time of Jack (time in hours). Then $rt = (3/4)(r)(t + 1/9)$ gives $t = 1/3$.

31. (d) $f_{35} = f_{28}(f_2(f_5)$ and $f_{35} = f_5$ gives $f_{28} = (f_2)^{-1}$. From $f_2 = \frac{2f_1-1}{f_1+1} = \frac{x-1}{x}$ it follows that $(f_2)^{-1} = \frac{1}{1-x}$.

32. (c) Express as ($\frac{100a+10b+c}{a+b+c} = 10\frac{10a+b+c/10}{a+b+c}$. The fraction expression is > 1. Increasing a increases the terms in the numerator and denominator of the original fraction by a ratio of 100 to1 and thus increases the fraction; increasing b and c causes the ratios of 10,1 respectively and thus decreases the fraction. Setting $a = 1$, $b = c = 9$ gives 10 9/19.

33. (c) $2^{12} \equiv 1 \pmod{13}$ and if n is any positive integer then $(2^{12})^n \equiv 1^n \bmod (13) \equiv 1 \pmod{13}$. Thus $2^{1000} = (2^{12})^{83}(2^4) \equiv 16 \pmod{13} \equiv 3 \pmod{13}$.

34. (b) Let the three roots be r, $r + d$, $r - d$. Their sum is $3r$ and their product is $r(r^2 - d^2)$. Thus $3r = 9/4$ and $r(r^2 - d^2) = 15/64$. Solving gives $r = 3/4$ and $d = 1/2$ or $d = -1/2$.

35. (d) Write $P = (x^2 + 3x + 1) - 3(x - 10)$. Then $x = 10$ makes P a square. Suppose another value y for x makes P a square, and let $P = y^2$, $y > 0$. If $|x^2 + 3x + 1| < y$ then $3(10 - x) \ge 2y - 1 > 2|x^2 + 3x + 1| - 1$ which holds only for integers x from –6 to 2, and none of these make P a square.