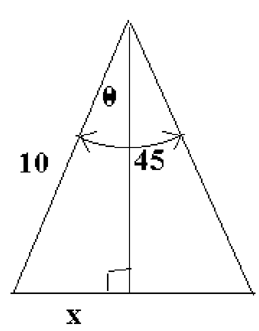
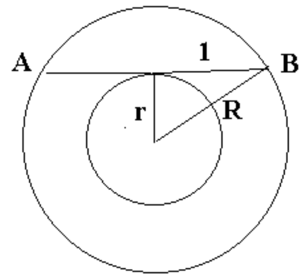
Answers and Brief Solutions to E1985

1. (b) Let $P(n)$ be the probability of exactly n heads; then $P(n) = \frac{4!}{n!(4-n)!}(1/2)^n$. The answer is $P(2) + P(3) + P(4) = (6 + 4 + 1)(1/16)$ or $1 - [P(0) + P(1)] = 1 - [(1 + 4)(1/16)]$.
2. (e) Let the first term be a and the ratio r. Then $a + ar = 18$ and $ar - a = 12$. Solving gives $a = 3$ and $r = 5$. The third term is ar^2.
3. (d) $A(N)$ is true for $N = 1,2,3,4,5,6$. $B(N)$ is equivalent to $(2N + 3)(N - 1) > 0$ which is false only if $N = 1$.
4. (c) The quotient is approximately $5/(-8x)$ when x is very large.
5. (c) Exams 1 and 2 scores give a deficiency of $(92 - 82) + (92 - 90) = 12$ points. Thus she must average $12/3 = 4$ points more than 92 on the next exam.
6. (b) The probability both numbers are the same is 1/10; thus the answer is $(1/2)(1 - 1/10)$.
7. (a) Let S,F be the original number of students and faculty. Then $S/F = 30$. The new ratio is $S/(1.2F) = 30/1.2$.
8. (c) By similarity $x/2 = 5/x$ gives $x = 4$ and $17/AB = 4/2$ gives $AB = 17/2$.
9. (c) Let D be the trip distance and t_1 and t_2 the two times. Then $D/t_1 = 40$ and $2D/(t_1 + t_2) = 50$. Elimination of t_1 gives $D/t_2 = 200/3$.
10. (d) The contrapositive of the given statement is "If not (B or not C)then A"; also "not (B or not C)" is equivalent to "not B and C"..
11. (d) Use the binomial approximation $(1 + x)^n \cong 1 + nx + \frac{n(n-1)}{2}x^2$. Note for example $(1.001)^{100} - 1.1 \cong (50)(99)/1000^2 < .005$ and $(.9)^{-1} - 1.1 \cong .01$.
12. (b) Possible prime divisors are 2,3,5,7,11,13. The integers are $2\times3\times5$, $2^2\times3\times5$, $2\times3^2\times5$, $2\times3\times7$, $2\times3\times11$, $2\times3\times13$, $2\times5\times7$, and $2^2\times3\times7$
13, (a) By geometry it is an intersection point of the circle $x^2 + 2x + y^2 = 4$ and the line $y = -x$. Solving $y^2 - 2y + y = 4$ gives $y = 2$.
14. (d) Each repetition reduces the concentration by 1/2. The result is $50(1/2)^{10} \cong 50/1{,}000 = .05$.
15. (c) Let d_j, d_b, r_j, r_b, t_j, t_b, respectively denote distances, rates,times of Joe and Bill. Then $d_j = 2d_b$, $r_j = r_b + 10$, $t_b = 6$, $t_j = 5$, $d_j = r_j t_j$, $d_b = r_b t_b$, give $r_b = 50/7$.
16. (c) From the given conditions $13 = a_3 + a_2$ and $a_3 = a_2 + 1$ yield $a_2 = 6$, $a_3 = 7$. Thus $a_5 = 13 + 7 = 20$ and $a_6 = 20 + 13 = 33$.
17. (a) Let C be Jack's cost. Then $C = P(1 + r/100)$ and $Q = C(1 - s/100)$.
18. (a) If A is the amount then $A(1 + .12/4)^{4\text{x}10} = 1{,}000$.
19. (b) Letting ^ denote exponent then $x = 6^{\log_6 x} = 6^{\log_6 a+1} = 6^{\log_6 a}(6^1) = 6a$ and $2^{\log_a x} = 8 = 2^3$ one may conclude $\log_a 6a = 3$. Thus $a^3 = 6a$.
20. (e) (see the Figure below) The third side is $2x$ where $x = 10 \sin(45°/2)$. Letting $\theta = 45°/2$ then $\sin 2\theta = 2 \sin\theta \cos\theta$ gives $\sqrt{2}/2 = (2\sin\theta)\sqrt{1-\sin^2\theta}$ and solving gives



$\sin^2\theta = (2 - \sqrt{2})/4$.

21. (a) From the figure below $R^2 - r^2 = 1$ and the area $= \pi R^2 - \pi r^2$



22. (d) If the first equation is multiplied by 3 and the second equation is subtracted from that result, the equation obtained is $2x + 4y + z = 0$. Comparing with the third equation, if $t \neq 0$ then there are no solutions, and if $t = 0$ there are infinitely many solutions.

23. (d) For any positive integer K there are two values of N, namely $N = 2K$ and $N = 2K + 1$ such that $f(N) = K$. It follows that for all positive integers P and K there are 2^P values for N which satisfy $T^P(N) = K$.

24. (e) If r,s,t are the roots then the equation is $(x - r)(x - s)(x - t) = 0$ and this gives $x^3 + (-r - s - t)x^2 + (rs + rt + st)x - rst = 0$. Thus $A = -S$ and $C = -P$.

25. (e) The successive numbers are $(a + 1)^2 = 4$, $(4 + 1)^2 = 25$ and $(25 + 1)^2 = 676$.

26. (c) Let a *Tenset* mean a sequence of 10 consecutive two or three digit numbers, the last two being in one of the ranges 01-10, 11-20, 21-30, etc. Then each *Tenset* has at most one number whose digits total 10, and exactly one provided the sum of the hundreds and tens digits do not exceed 10 (the one exception is the *Tenset* 001-010). Thus the count includes 9 numbers less than 100, 10 in the 100's, 9 in the 200's, 8 in the 300's, 7 in the 400's, etc. The sum is 9 + 10 + 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2.

27. (a) $N \bmod 100 \equiv bc$ and $bc \bmod 10 \equiv c$.

28. (a) Let $x^{70} + x^{10} + x - 5 = (x - 1)Q(x) + Ax + B$. Then $x = 1$ gives $-2 = A + B$ and $x = -1$ gives $-4 = -A + B$. solve for A and B.

29. (b) From $(2N)^2 = 4N^2$ and $(2N + 1)^2 = 4N^2 + 4N + 1$ it is seen that division of a perfect square by 4 gives a remainder of 0 or 1. Thus division of $100N + 11$ by 4 gives a remainder of 3.

30. (a) The last digit of successive powers of 3, starting with the first power, are 3,9,7,1,3,9,7,1,… . Thus the last digit of 3 is 3 if $N \bmod 4 \equiv 2$, 7 if $N \bmod 4 \equiv 3$ and 1 if $N \bmod 4 \equiv 0$. Since $1000 \bmod 4 \equiv 0$ the conclusion follows.

31. (b) $S_N = 2(1 + 2 + \ldots + N) = N(N + 1)$. Since $N^2 < S_N < (N + 1)^2$, then S_N cannot be a perfect square.

32.(a) The case $M = 2$ and $N = 2$ eliminates *II*; the case $M = 3$, $N = 1$ eliminates *IV*. The pairs (M,N), $(N,M + 2N)$, $(N,M - N)$ all have the same set of common divisors.