Answers and Brief Solutions to E1987

1. (c) Let $F$ be the score on the first exam and $d$ the difference on successive exams. Then $F + d/2 = 78$ and $F + d = 80$.

2. (a) The numerator is near 1 and the denominator is negative and near 0.

3. (d) Subtract equation 2 from twice equation 1 to get $-y - 3z = 7$ and add $-2$ times equation 1 to equation 3 to get $-y - 3z = t - 16$. The answer follows from $7 = t - 16$.

4. (d) This is the number of solutions of the equation $i + j + k = 6$ where $i,j,k$ are non-negative integers. For $i = 0,1,2,3,4,5,6$ there are $7,6,5,4,3,2,1$ solutions.

5. (e) $0.3(x + y) = y$ and $0.6(x + y + z) = y + z$; eliminate $x$ and simplify.

6. (a) $y = f(x - 2)$ has the point $(5,7)$ and $y = 3f(x - 2)$ has the point $(5,21)$.

7. (a) Substitution of $y = x + B$ gives $x^2 + 2x + (x + B)^2 = 0$. This is a quadratic equation in $x$ and setting the discriminant equal to 0 gives $B^2 - 2B - 1 = 0$. Solve for $B$.

8. (b) By the binomial theorem, $(1 + x)^{1/2} \approx 1 + x/2$ if $x$ is small.

9. (d) There are $3^3 = 27$ possible outcomes and $3! = 6$ of these produce all three balls.

10. (d) $60 \text{ mod } 31 \equiv 29$, $29 \text{ mod } 11 \equiv 7$ and $46 \text{ mod } 7 \equiv 4$.

11. (b) Let $d$ be the distance from $A$ to $B$. Then the total distance driven is $3d$ and the total time is $d/10 + d/40 + d/50$. The average speed is the total distance divided by the total time.

12. (e) If $n$ is the number of pens and $c$ the cost then $n(1.05)c = (3 + n)c$.

13. (b) Summing the number of points whose $x$ coordinates are 1, 2, 3, …, 17 gives $9 + 8 + 8 + 7 + 7 + … + 1 + 1$.

14. (c) If $P$ is the amount of the investment then $3P = P(1 + r/2)^{20}$.

15. (a) The numbers are in succession the following powers of 2: 32, 48, 24, 128, 48. Thus the answer is $2^{128-24}$.

16. (d) Let $r,s$ be the roots. Then $m = -(r + s)$ and $n = rs$. If $n$ is odd then each of $r$ and $s$ is odd and $m$ is even.

17. (e) If he wins the last two bets then he wins $2$.

18. (d) By II, $Q$ is true and $P$ is false; only (d) is true in this case.

19. (a) Let $d$ be the number of feet Bill runs; then Tom runs $d - 100$ feet and hence $10/9 (d - 100) = d$.

20. (d) If $x$ is the side opposite the $75^\circ$ angle then, by the law of sines, $x/\sin 75^\circ = 6/\sin 60^\circ$. Apply $\sin 75^\circ = \sin(45^\circ + 30^\circ) = \sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ$.

21. (c) $I$ is true since $S = a - (b - c) - d$ and III is true since $S = (a - b) + (c - d)$.

22. (a) $\log_2 3 = 1/A$ gives $\log_2 3 = 3/A$ and $\log_2 5 = B$ gives $\log_2 5 = 4B$; add $3/A + 4B$.

23. (b) $F(1) = F(2) = 1$; $F(3) = 0$; $F(4) = F(5) = -1$; $F(6) = 0$; $F(7) = F(8) = 1$ implies $F(n) = F(n - 6)$ for $n > 6$. Thus $F(1,000) = F(4)$.

24. (e) **Method I**: By a counting argument there are successively 1, 8, 21, 20, 5 words with 0, 1, 2, 3, 4 a’s. **Method II**: Let $x_n$ be the number of $n$ letter words using only $a,b$ without two consecutive $a$’s; then $x_n = x_{n-1} + x_{n-2}$ and $x_1 = 2; x_2 = 3$.

25. (c) The region is a rectangle bounded by the lines $y = x + 3$, $y = x - 3$, and $y = 2 - x$, $y = -2 - x$. The vertices are $(5/2,1/2), (1/2,-5/2),(-5/2,1/2),(-1/2,5/2)$. The distances between opposite sides are $3 \sqrt{2}$ and $2 \sqrt{2}$.

26. (e) $x^2 + x + (1 - A) = 0$; using the quadratic formula set the discriminant $1 - 4(1 - A) = 0$. 
27. (c) 1987 = 87x22 + 19x3 + 16. Each reduction of the multiplier of 87 by 1 causes an increase in the multiplier of 19 by 5 and a decrease in the remainder by 8 since 87 = 19x5 – 8. Thus 1987 = 87x20 + 19x13.

28. (c) 9! ≈ 3.6x10^5; multiply this by 10^{11}x1.1x1.2x…1.9x2 ≈ 6.7x10^{12}

29. (b) Starting with 1 as the smallest integer and increasing successively by 1 there are 10,9,9,8,8,7,7,…,1 possibilities.

30. (d) s = rx and c = 2r sin(x/2); thus s/c = x/(2 sin(x/2)).

31. (e) x/1 = 4/y = z/5 gives xy = 4 and yz = 20. Values x = 4, y = 1, z = 20 give the maximum.

32. (c) A congruent triangle may be placed in the xy plane with vertices (0,0), (0,2), (1,0). If (x,y) is the point common to the rectangle and hypotenuse then y = -2x + 2. Thus the area of the rectangle is A = x(-2x + 2) = -2(x – 1/2)^2 + 1/2.