ANSWERS AND BRIEF SOLUTIONS TO E1989

1. (d) The total score for the honor students is 80x30 = 2400 and for the math students is 86x16 = 1376. The total score for all is (30 + 16 – 10)x81 = 1376. The total duplicate score is 2400 + 1376 – 2916 = 860 for 10 students.

2. (a) If \( x > 0 \) the equation \( x^2 – 7x + 3x = 0 \) gives \( x = 0, 4 \). If \( x \leq 0 \) the equation \( x^2 + 7x + 3x = 0 \) gives \( x = 0, -10 \); the answer is 4 + (-10).

3. (c) \( S = 7(2 + 3 + 4 + \ldots + 28) = (7)(28)(29)/2 – 7. \)

4. (b) \( 9! = 362,880 \) and \( 10! = 3,628,800. \)

5. (c) \( b \cdot a = b + 2a \) and \( a \cdot b = a + 2b + 2c \) and \( a \cdot (b \cdot c) = a + 2b + 4c; \) \( (a \cdot b) \cdot c = a + b + c - 4 = a \cdot (b \cdot c) \)

6. (d) If \( a \) is the first term and \( r \) the ratio than \( ar^2 = 3 \) and \( ar^6 = 48. \) Thus \( r = 2 \) and \( a = 3/4. \)

7. (d) Of any three successive integers one must be divisible by 2 and one (perhaps the same) by 3; hence the product is divisible by 6.

8. (c) Multiply the numerator and denominator by \( x; \) if \( x \) is small then the new numerator is near 5 and the new denominator is near –2.

9. (e) \( (5/6)(2/3) = 5/6 \)

10. (c) The logic principle is ‘if \( p \) then \( q \)’ is equivalent to ‘(not \( p \)) or \( q \).’

11. (a) The third equation is the second equation minus the first equation so the equations are dependent. If \( y \) is eliminated from any two equations the result yields \( x = 3z + 4. \)

12. (b) The final mixture has \( [(20)(.2) + (10)(.3)](x/30) + (10)(.4) = 7x/30 + 4 \) ounces of alcohol and \( x + 10 \) total ounces. Solve \( 7x/30 + 4 = (.3)(x + 10). \)

13. (a) The altitudes meet \( 2/3 \) of the distance from the vertex; thus the altitude length = \( 3/2 \) and the area is \((2)(1/2)(3/2)(3\sqrt{3}/2)\)

14. (b) The exponents of \( x \) add to \( 8 \) with the combinations \( 3,3,2,0; 3,3,1,1; 3,2,2,1; 2,2,2,2 \) respectively with \( 12,6,12,1 \) possibilities.

15. (d) \( (x + i)(x – i) = x^2 + 1 \) divides the left side of the equation, and the quotient is \( (2x - 3)(x + 4); \) thus \( 3/2 \) and \(-4 \) are the real roots.

16. (d) After cancellation of the common terms the sum is \( 1 + 1/2 - 1/10 - 1/11. \)

17. (b) By the binomial expansion, if \( x \) is small then \( (8 + x)^{1/3} \approx 8^{1/3} + (1/3)(8^{-2/3})(x). \)

18. (e) \( \text{Substitute } x = .0036 \) to obtain the answer.

19. (d) Solve \( \log x + 2 \log y = 4, \log y – \log z = 3 \) and \( \log x + 3 \log z = -4 \) to get \( x = 4, y = 2 \) and \( z = 1/4. \)

20. (e) Adding the probabilities of the favorable cases black, red and red, red gives \( (2/5)((3/5) + (3/5)(2/4). \)

21. (b) \( \cos 2x = 3 \cos^2 x – 1 = 1/4 \) gives \( \cos^2 x = 5/8; \) \( \cos 2x = 1 – 2 \sin^2 x = 1/4 \) gives \( \sin^2 x = 3/8; \) \( \tan^2 x = \sin^2 x/\cos^2 x \) gives the answer.

22. (e) The values of \( f(n) \) are respectively \( 1, 2, 1/2, 1, 0, 0, undefined \) since \( f(5) = 0 \) and \( \log 0 \) is undefined. Therefore \( f(n) \) is undefined if \( n > 5. \)

23. (e) Bill can only break even if he wins the last two bets; John wins $9 if Bill wins the first two bets.

24. (c) \( (1 + x)(1 – y) – 1 = x – y – xy. \) In (c), (d) the largest of the \( x, y \) terms is less than in the other cases; choose (c) since then \( xy \) and \( x – y \) have the same sign.
25. (a) \( f(50) = 2/7 + 1 = 9/7 \).
26. (d) The longest side of the triangle is \( 21 \) or \( x \). If \( 21^2 < 9^2 + x^2 \) and \( x^2 < 21^2 + 9^2 \), then all angles are acute; this is true for \( x = 19, 20, 21 \), and 22.
27. (a) If 3 divides \( n + 2 \) then 3 also divides \((n + 2) + 3k\) for all integers \( k \); thus 3 divides \( n - 1 \) which is a factor of \( n^2 + 6n - 7 \).
28. (c) If \( r \) is the annual rate of interest then \( A^{10} = 2 \) and \( 10 \log A = \log 2 \) where \( A = (1 + r/365)^{365} \); solving \( A^x = 3 \) gives the answer.
29. (a) After the first four terms in the sum, each term is divisible by 5 and the sum of the first four terms is 35 which is also divisible by 5.
30. (a) Letting \( x = \angle BCD \) then also \( x = \angle ABE \). From triangle \( BCD \) it is seen that \( \tan x = \frac{3}{2} \) and hence \( \cos x = \frac{2}{\sqrt{13}} \); also \( \cos x = BE/4 \) from triangle \( ABE \). Therefore \( BE/4 = \frac{2}{\sqrt{13}} \).