Answers and Brief Solutions to E1990

1. (a) Solve the equations $(x + y + z)/3 = 80$; $(x + y)/2 = 76$; $x = 10 + y$.
2. (a) $[x - (1 - i)][x - (1 + i)]$ is a factor of $P(x)$.
3. (e) $m = 16$, $n = 28$.
4. (d) $(n + 1) + (n + 2) + \ldots + (n + k) = k\dfrac{n + (k+1)}{2}$ and $(k + 1)/2$ is an integer if and only if k is odd.
5. (d) If S is the current salary and D the current deductions then $S - D = .7S$ is the previous net amount and $1.1S - (1.15)(.3S) = .755S$ is the new amount. The answer follows from $(.755 - .7)/.7 \approx .08$.
6. (a) 7x2 + 1 = 15. Each team loses 2 games except the winner which loses 1 game.
7. (d) By the Euclidean Algorithm 8547/4810 leaves a remainder of 3737; 4810/3737 leaves a remainder of 1073; 3737/1073 leaves remainder of 518; 1073/518 leaves a remainder of 37; 518/37 leaves a remainder of 0. Thus 37 is the greatest common divisor.
8. (c) The line segments divide the original triangle into 4 congruent triangles.
9. (c) If f is the initial bet then $f + (f + 1) + (f + 2) + (f + 3) + \ldots + (f + 29) = (f + 30) + \ldots + (f + 49)$ gives $30f + (29\text{x}30)/2 = 20f + (49\text{x}50)/2 - (29\text{x}30)/2$.
10. (c) If $x < 2$ then no conclusion can be drawn form I; if $y > 9$ then I implies $x > 3$ is false.
11. (c) The following is the list of primes less than 50, each followed by the number of smaller primes which gives a product < 100: 2(0), 3(1), 5(2), 7(3), 11(4), 13(4), 17(3), 19(3), 23(2), 29(2), 31(2), 37(1), 41(1), 43(1), 47(1); these total 30.
12. (b) If a is the first term and r the ratio then $a + ar^2 = 2$ and $ar = a - 1$. Solving for a gives $a = 1 + \sqrt{2}/2$ or $a = 1 - \sqrt{2}/2$.
13. (d) $a/b < c/d$ is equivalent to $ad < bc$. Similarly $\dfrac{a+3c}{b+3d} < c/d$ is also equivalent to $ad < bc$ as is $a/b < \dfrac{a+3c}{b+3d}$.
14. (c) Add $-a$ times the last equation to the first, and $-d$ times the last equation to the second. Then $(b - a)y = c - a$ and $(e - d)y = -d$; equate the solutions for y.
15. (e) $\cos x \cos y + \sin x \sin y = \sqrt{6}/3$, $\sin y = \sqrt{2}/\sqrt{6}$, $\cos y = 2/\sqrt{6}$ gives $\sqrt{2}/2 \sin x = 1 - \cos x$; square both sides of the equation and use $\sin^2 x = 1 - \cos^2 x$.
16. (b) The probability exactly one number occurs is 1/36 and that exactly three occurs is (5/6)(4/6); subtract their sum from 1.
17. (d) If d is the distance from A to B and t the total time then $t = d/r + d/2r + d/3r$ and the average rate is $3d/t$.
18. (e) If the square has sides of length x then the inscribed circle has radius $x/2$ and its inscribed square has area $x^2/2$. Thus each pair of inscribings multiplies the area by ½.
19. (c) $\log_3(\log_2 w) = 2$; $\log_2 w = 9$ and $w = 512$.
20. (a) The shifts give $y - 4 = 3(x - 2) + 5$; for the rotation replace x by $-y$ and y by x.
21. (c) If x is the unknown rate then $(1 + r/4)^4 = (1 + x/2)^2$

22. (b) There are (10x9x8)/(3x2x1) total combinations. Of those which add up to 8 or less, if the largest number is 7 there is 1, if 6 then 2, if 5 then 4, if 4 then 5, if 3 then 1 and if 2 then 1; these total 16.

23. (c) Successive squaring gives approximately for powers of 3: 1-3; 2-9; 4-80; 8-6.4x10^{3}; 16-4x10^{7}; 32 –1.6x10^{15}; 64-2.6x10^{30}. Multiply powers of 4, 32, 64 to get approximately 3.2x10^{47}. Alternatively $3^2 < 10$ implies $3^{100} < 10^{50}$ and $3^5 > 100$ implies $3^{100} > 10^{40}$; by elimination 47 is only possible answer.

24. (d) $f(2) = 4 - 1$; $f(4) = 4 - 1/2$; $f(6) = 4 - 1/4$, …, $f(2n) = 4 - 1/2^{n-1}$; set $n = 50$.

25. (b) Let P be the point of tangency above the x axis. Then the triangle formed by $(a,0)$, P and the origin is an isosceles right triangle with a leg of length 1; the hypotenuse is $\sqrt{2}$.

26. (a) Amounts of solution are: Initially A: $r/50$ and B: $s/50$; after the first pouring A: $\frac{r}{100}$ and B: $\frac{2s+r}{100}$; after the second pouring A: $\frac{r}{100} + \frac{2s+r}{300} = \frac{4r+2s}{300}$. The answer is $\frac{4r+2s}{300}$x(1/2)x100.

27. (b) Let $f^2(x) = (x + 3) + 3 = x + 6$; $f^3(x) = (x + 3) + 6 = x + 9$; etc.

28. (b) Let $x + y - z = -M$, $x - y - z = S$ where M is a large positive and S a small positive number. Then $x = \frac{S+1}{2}$, y = $-\frac{M+S}{2}$, z = $\frac{M+1}{2}$.

29. (a) By the binomial expansion the difference is approximated by $10 \text{x} (10^{10})^9$ x 1

30. (e) The expansion includes 10 products having one a^3 and nine 1's, $\frac{2x10x9}{2x1}$ products having one a, one a^2 and eight 1's, and $\frac{10x9x8}{3x2x1}$ products having three a's and 7 1's. Adding these numbers gives 10 + 90 + 120.