Answers and Brief Solutions to E1992

1(b) If \( x \) is the number of quizzes then 
\[ 180+280+40x = .78(600+50x) \]

2(a) If \( b \) is the number of boys and \( g \) of girls then 
\[ \frac{b}{g} = \frac{5}{3} \text{ and } \frac{1.2b}{1.3g} = \left(\frac{1.2}{1.3}\right)(\frac{5}{3}) \]

3(c) Each roll the probability of a 2 or 5 is \( 1/3 \); answer is \( (1/3)^4 \)

4(d) The given statement is equivalent to \( y \geq b \) or \( z \geq b \); the negation is equivalent to 
\( y < b \) and \( z < b \).

5(a) By appropriate addition and subtraction of equations one can obtain 
\[ 0 = c-d+a-b \] The original system is then equivalent to 3 equations in 4 unknowns which has infinitely many solutions.

6(d) Let \( d \) be the distance. Then 
\[ \frac{d}{2r} + \frac{d}{2(3r/4)} = \frac{d}{2s} + \frac{d}{2(5s/4)} \]

7(c) Tom wins with the sequences of winners and associated probabilities 
\( TT(1/4), TBT(1/8), BTT(1/8) \) and these add to 1/2.

8(d) The amounts of water and alcohol at the various stages are: 
\( [9,1];[9,2];[(90/11),(20/11)];[(90/11),(31/11)] \)
(we may ignore the last removal). The answer is \( \frac{31}{31+90} \times 100 \).

9(e) \( 3+5+7 \) is odd and not a prime; \( 3+5+11 \) is a prime; \( 2+3+5 \) is even

10(b) If \( L \) is the low value and \( H \) the high value of the possible numbers then Mary guesses 
\[ \frac{L+H}{2} \] if \( L+H \) is even and 
\[ \frac{L+H+1}{2} \] if \( L+H \) is odd. Then Mary's incorrect guesses reduce successively from 100 to 50, 25, 12, 6, 3, 1 possible numbers.

11(d) These are the numbers which are the squares of multiples of 3 i.e. squares of 
\( 3, 6, 9, 12, 15, 18, 21 \).

12(a) Since \( \sin A = \sin B = 4/5 \) it follows that \( a=b=5/2 \) and the altitude divides \( c \) into two segments of length 3/2 each.

13(a) The diagonals of \( AB \) are diameters of the circle, and have length 2. Thus the length of the sides of \( ABCD \) are \( \sqrt{2} \), the area of \( ABCD \) is 2 and triangle \( CDE \) has half this area.

14(e) Solve the equations simultaneously to get 
\[ (x - 1)^2 + x = r^2 \] which is a quadratic equation in \( x \). The discriminant is less than 0 if \( r^2 < 3/4 \).
15(e) $2 \log_3 (xyz) = \log_3 (x^2 y^2 z^2) = \log_3 27 = 3$.

16(b) The quotient is near $4y/(-2y)$

17(c) For $x = 1$ there are 8 solutions; for $x = 2$ seven solutions; ..., for $x = 8$ one solution. Answer is $8 + 7 + 6 + 5 + 4 + 3 + 2 + 1$.

18(b) Form a triangle $OPQ$ where $O$ is the center of the circle and $P, Q$ are points on the circle, and $A$ is the angle $POQ$. Draw an altitude $PR$ from $P$ to $OQ$. Then $PR = \sin A$, $QR = 1 - \cos A$, and $(PQ)^2 = (PR)^2 + (QR)^2$. Thus $x = \sin^2 A + (1 - \cos A) = 2(1 - \cos A)$

19(e) $ABC = 36A + 6B + C$ is divisible by 4 only if $6B + 4$ is divisible by 4 which is not necessarily true in any of (a)-(d).

20(e) $P(1+.08/4)^4 = P(1.02)^4 \approx P[1+4(0.02)+6(0.02)^2]$ (Binomial Theorem)

21(d) $6 = 1 + A + B + C$ and $0 = 8 + 4A + 2B + C$. Division of $f(x)$ by $x - 2$ gives $x^2 + (A + 2)x + (B + 4A + 4)$. The sum of the other two roots is $-(A + 2)$. Hence $A = -3, B = -4, C = 12$.

22(c) The simultaneous solutions of the two inequalities is represented graphically by the square with vertices (-2,0), (0,-2), (2,0), (0,2). The maximum $xy$ occurs along the line $y = 2 - x, 0 \leq x \leq 2$. The maximum value of $x(2 - x) = 1 - (x - 1)^2$ is 1.

23(c) By the binomial theorem $1000^{1/3} = (1000 + 1)^{1/3} \approx 10 + (1/3)(1000)^{2/3} - (2/9)(1000)^{-2/3} = (10 + 1/300) - 2/900,000$

24(c) If $N = 4$ then $M^2 - 16 \equiv 1 \mod 10$ implies $M^2 \equiv 7 \mod 10$ which is not possible for any integer $M$.

25(b) From the sequence 1, 3, 3/2, 7/2, 7/4, 15/4, ..., it is seen $a_{2n-1} = \frac{2^{n+1} - 1}{2^{n-1}}$ and $a_{2n+1} = \frac{2^{n+2} - 1}{2^n}$. Thus $a_{99} = (2^51 - 1)2^{49}$ and $a_{101} = (2^52 - 1)2^{50}$.

26(a) This may be seen by sketching the graphs of $y = x + 1, x \geq 2; y = -x + 5, x \leq 2$ and $y = x - 3, x \leq 5$ and $y = 7 - x, x \geq 5$.

27(e) For example if $k$ is a positive integer then $k^2, (2k)^2, (4k)^2, ...$ is such a geometric sequence.
28(a) The number must also divide $2(3n + 17) - 3(2n - 8) = 58$.

29(a) By substitution $y = c$ is also a root. By long division the function may be factored to $(y - c)(y^2 + cy - 1)$ and the sum of the roots of the quadratic factor is $-c$.

30(d) $S \geq \frac{1}{x^2}$ if $\frac{1}{x^3} + \frac{1}{x^2} \geq \frac{2}{x^3}$. Multiply by $x$ and simplify to $(x - 1)^2 \geq 0$. 

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