Answers and Brief Solutions to E1992

1(b) If x is the number of quizzes then $180+280+40x = .78(600+50x)$

2(a) If b is the number of boys and g of girls then $b/g=5/3$ and $1.2b/1.3g = (1.2/1.3)(5/3)$

3(c) Each roll the probability of a 2 or 5 is 1/3;answer is $(1/3)^4$

4(d) The given statement is equivalent to $y \geq b$ or $z \geq b$; the negation is equivalent to $y < b$ and $z < b$.

5(a) By appropriate addition and subtraction of equations one can obtain $0 = c-d+a-b$.The original system is then equivalent to 3 equations in 4 unknowns which has infinitely many solutions.

6(d) Let d be the distance. Then $\frac{d}{2r} + \frac{d}{2(3r/4)} = \frac{d}{2s} + \frac{d}{2(5s/4)}$

7(c) Tom wins with the sequences of winners and associated probabilities *TT*(1/4),*TBT*(1/8),*BTT*(1/8) and these add to 1/2.

8(d) The amounts of water and alcohol at the various stages are:

[9,1];[9,2];[(90/11),(20/11)];[(90/11),(31/11)]

(we may ignore the last removal). The answer is $\frac{31}{31+90}$ x 100.

9(e)3+5+7 is odd and not a prime; 3+5+11 is a prime; 2+3+5 is even

10(b) If L is the low value and H the high value of the possible numbers then Mary guesses $\frac{L+H}{2}$ if $L+H$ is even and $\frac{L+H+1}{2}$ if $L+H$ is odd. Then Mary's incorrect guesses reduce successively from 100 to 50,25,12,6,3,1 possible numbers.

11(d) These are the numbers which are the squares of multiples of 3 i.e. squares of 3,6,9,12,15,18,21.

12(a) Since $\sin A = \sin B = 4/5$ it follows that $a=b=5/2$ and the altitude divides c into two segments of length 3/2 each.

13(a) The diagonals of *AB* are diameters of the circle, and have length 2. Thus the length of the sides of *ABCD* are $\sqrt{2}$, the area of *ABCD* is 2 and triangle *CDE* has half this area.

14(e) Solve the equations simultaneously to get $(x - 1)^2 + x = r^2$ which is a quadratic equation in x. The discriminant is less than 0 if $r^2 < 3/4$.

15(e) $2 \log_3 (xyz) = \log_3 (x^2 y^2 z^2) = \log_3 27 = 3$.

16(b) The quotient is near $4y/(-2y)$

17(c) For $x = 1$ there are 8 solutions; for $x = 2$ seven solutions; …, for $x = 8$ one solution. Answer is 8+7+6+5+4+3+2+1.

18(b) Form a triangle OPQ where O is the center of the circle and P,Q are points on the circle, and A is the angle POQ. Draw an altitude PR from P to OQ. Then $PR = \sin A$, $QR = 1 - \cos A$, and $(PQ)^2 = (PR)^2 + (QR)^2$. Thus $x^2 = \sin^2 A + (1 - \cos A) = 2(1 - \cos A)$

19(e) $ABC = 36A+6B+C$ is divisible by 4 only if $6B+4$ is divisible by 4 which is not necessarily true in any of (a)-(d).

20(e) $P(1+.08/4)^4 = P(1.02)^4 \approx P[1+4(.02)+6(.02)^2]$ (Binomial Theorem)

21(d) $6=1+A+B+C$ and $0=8+4A+2B+C$. Division of $f(x)$ by $x - 2$ gives $x^2 + (A+2)x + (B+2A+4)$. The sum of the other two roots is $-(A + 2)$. Hence $A=-3$, $B=-4$, $C=12$.

22(c) The simultaneous solutions of the two inequalities is represented graphically by the square with vertices (-2,0),(0,-2), (2,0),(0,2). The maximum xy occurs along the line $y = 2 - x$, $0 \le x \le 2$. The maximum value of $x(2 - x) = 1-(x - 1)^2$ is 1.

23(c) By the binomial theorem $1000^{1/3} = (1000+1)^{1/3} \approx$
$10 + (1/3)(1000)^{-2/3} - (2/9)(1000)^{-5/3} = (10 + 1/300) - 2/900{,}000$

24(c) If $N = 4$ then $M^2 - 16 \equiv 1 \text{ Mod } 10$ implies $M^2 \equiv 7 \text{ Mod } 10$ which is not possible for any integer M.

25(b) From the sequence 1,3,3/2,7/2,7/4,15/4,... it is seen $a_{2n-1} = \dfrac{2^{n+1}-1}{2^{n-1}}$ and $a_{2n+1} = \dfrac{2^{n+2}-1}{2^n}$. Thus $a_{99} = (2^{51} - 1)/2^{49}$ and $a_{101} = (2^{52} - 1)/2^{50}$.

26(a) This may be seen by sketching the graphs of $y = x+1$, $x \ge 2$; $y = -x + 5$, $x \le 2$ and $y = x - 3$, $x \le 5$ and $y = 7 - x$, $x \ge 5$.

27(e) For example if k is a positive integer then $k^2, (2k)^2, (4k)^2$, ... is such a geometric sequence.

28(a) The number must also divide $2(3n + 17) - 3(2n - 8) = 58$.

29(a) By substitution $y = c$ is also a root. By long division the function may be factored to $(y - c)(y^2 + cy - 1)$ and the sum of the roots of the quadratic factor is $-c$.

30(d) $S \geq 1/x^2$ if $1/x^3 + 1/x^2 \geq 2/x^3$. Multiply by x and simplify to $(x - 1)^2 \geq 0$.