Answers and Brief Solutions to E2001

1. (a) = 2*(2*(1/2)) = 2*(4/5) = 10/14

2. (d) If D is the diameter of the circle then D is the length of the sides of the larger square and is the length of a diagonal of the smaller square. Thus the sides of the smaller square have length $D/\sqrt{2}$ and $A/a = (\pi D^2/4)/(\pi D^2/8)$

3. (d) From $x^2 - 3x + 2 = (x - 1)(x - 2) < 0$ it follows that $1 < x < 2$

4. (d) Substitution of the given points into the parabola equation gives $1 = a + b$ and $7 = 4a + b$ from which $a = 2$ and $b = -1$. Thus the equation is $y + 1 = 2x^2$

5. (a) If x is the number removed then $x = 90N - 100(N - 1) > 0$ gives $100 > 10N$.

6. (e) Method 1; After a first sock is selected there are 7 socks remaining and 3 are of the same color as the first. Method 2: Letting C(n,r) denote the number of combinations of r objects from n there are C(8,2) = 28 possible choices of a sock pair and 2C(4,2) = 12 choices of a pair having the same color; thus the answer is 12/28.

7. (a) After the second transfer A has $4 – 16/x + 4(x – 4)/x$ and B has $(x – 4) – 4(x – 4)/x + 16/x$ gallons of juice; equating these and simplifying gives $(x – 8)^2 = 0$.

8. (b) If the numbers are a, ar and ar^2 then $a(1 + r + r^2) = 52$ and $a(r^2 - 1) = 32$. Then eliminating a gives $52(r^2 - 1) = 32(1 + r + r^2)$ which has $r = 3$ as the positive root. Substituting gives $a = 4$ and thus ar = 12.

9. (e) Let y be the unknown ; then $(1 + x/100)(1 – y/100) = 1$; solve for y in terms of x.

10. (c) The lines $y – x \leq 1$ and $y – x \geq -1$ intersect the lines $y + x \leq 1$ and $y + x \geq -1$ in the four points (1,0), (0,1), (-1,0), (0,-1) which form the vertices of a square whose sides have length $\sqrt{2}$.

11. (c) Solving for y and completing the square gives $y = (x - 4)^2 + 3$

12. (d) From $108 = 2^2 x 3^3$, $450 = 2x3^2x5^2$, and $405 = 3^4x5$ the least common multiple is $2^2x3^4x5^2$ x100 = 810,000.

13. (e) The area of the parallelogram is the product of the base and the height. Either the base is 10 and the height $5 \sin 60° = 5\sqrt{3}/2$ or the base is 5 and the height $10 \sin 60° = 5\sqrt{3}$.

14. (a) This is the probability that there is either 0 or 1 head among the first 4 tosses. The probability of 0 heads is $(1/2)^4$ and of 1 head is $4(1/2)^4$; add these values.

15 (a) Let x be the larger integer. Then $x^2 - (x - 1)^2 = 1999$ gives $2x = 2000$, or $x = 1000$.

16. (b) Let r_1 and r_2 be the rates of cars *1* and *2* respectively and *t* the unknown time. If *d* is the distance between *A* and *B* then $d = (r_1 + r_2)t = 4r_1 = 6r_2$ gives $r_1 = 3/2\ r_2$ from which $5/2\ r_2\ t = 6r_2$ and hence $t = 6/(5/2)$.

17. (c) $x = (97 - 3y)/2$. If *y* is odd then $97 - 3y$ is even and *x* is then an integer; if *y* is even then *x* is not an integer. There are 16 odd integers between 1 and 31.

18. (b) $2000 = 3(8)^3 + 7(8)^2 + 2(8)^1 + 0(8)^0$; thus $3 + 7 + 2 + 0 = 12$

19. (d) $2 \log_{10}100{,}000 = 2\log_{10}(10^5) = \log_{10}(10)^{10} = 10$ and $\log_{10}10 = 1$

20. (e) If *P* is the initial amount and *r* is the annual rate of interest then the value after 10 years is $2P = P(1 + r)^{10}$ from which $(1 + r) = 2^{1/10.}$ After 6 years the value is $P(1 + r)^6 = P(2^{1/10})^6 = P(2^{3/5})$.

21. (b) Using $(a + b)^{1/2} \approx a^{1/2} + 1/2\ a^{-1/2}b$, $4.004^{1/2} \approx 2 + 1/2\ (1/2)(.004) = 2.001$

22. (c) Let *T* be the total number of children, and *C, P, S* the number of coke, pepsi and sprite drinkers. Then $6 = T – C – P – S = \mathrm{T} – 1/2\ T – 1/3\ T – 1/3(1/2\ T – 1/3\ T)$. Solve for *T*.

23. (e) $x = 9$, $y = 5$, $z = 2$ makes all of *I,II III* false.

24. (c) The number of 0's at the end of 100! is the largest value of N such that 10^N divides 100! Since the prime factors of 10 are 2 and 5 then N is the lesser of the number of times (i) 2 (ii) 5 occurs as a factor of the product 100!. The number 5 occurs exactly once as a factor of the 20 multiples 5,10,15,…,100 of 5 excepting the 4 cases 25,50 ,75, 100 where it occurs twice giving a total of 24 times. The number 2 occurs as a factor at least once in each of the 50 even numbers between 2 and 100..

25. (e) Method I: $x^3 = x^2 - x = -1$ by algebra. Method II: By the quadratic formula $x = 1/2$ +/- $\sqrt{3}/2\ i = \cos 60° +/- \sin 60°\ i$. Then by DeMoivre's Theorem $x^3 = \cos 180°$ +/- $\sin 180°\ i$.

26. (d) By combining fractions $y = \dfrac{1}{x+1}$ and from $9/10 < x < 11/10$ it follows that $19/10 < x + 1 < 21/10$; taking reciprocals reverses the order of the inequalities.

27. (d) Method I: If *r,s* are the other two roots then $r + s + 2 = -(-4)$ and $2rs = 30$. Solving gives –3, 5 as values of *r,s*. Method II: $P(2) = 22 + 2A = 0$ gives $A = -11$. Division of $x^3 – 4x^2 + -11x + 30$ by $x – 2$ gives $x^2 - 2x – 15 = (x – 5)(x + 3)$ which has roots –3,5.

28. (c) Note $x_{n+1} = x_n - x_{n-1.}$ The sequence is 2,1,-1,-2,-1,1,2,1,… . Thus the sequence repeats every 6 terms and from 100 mod 6 = 4 the answer is the fourth term in the sequence which is -2.

29. (e) The number of *N* divisible by 6 is the integer value of 1000/6 and the number divisible by 8 is the integer value of 1000/8. These overlap by the integer value of 1000/24, since 24 is the least common multiple of 6 and 8. Thus the answer is 166 + 125 – 41 = 250.

30. (a) The amounts lost after succesive bets are A/2 , 2A/3, 3A/4, … giving the pattern nA/(n + 1) and hence after 10 bets 10A/11 has been lost leaving A/11. This is verified from n/(n + 1) = 1/n – 1/(n + 1) and hence the total sum amount lost is [(1-1/2) + (1/2 – 1/3) + (1/3 – 1/4) + … + (1/10 – 1/11)]A = [1 – 1/11]A.