Answers and Brief Solutions to E2002

1. (d) $x^2 = (10/100)x$ and $y^3 = (9/100)y$ gives $x = 0.1$ and $y = 0.3$

2. (c) The total number of points scored was 6x5 + 8x10 + 12x15 = 290 so the average is 290/30 = 29/3

3. (a) The third angle is 30°. Construct an altitude to a side of length 10; the length of the altitude is 10 sin 30° = 5. Since the corresponding base has length 10, the area is (1/2)(10)(5) = 25.

4. (b) The number of boys who play both baseball and football equals 1/3 *F* and also 2/5 *B*; thus 1/3 *F = 2/*5 *B.*

5. (e) The fourth term is the sum of the first four terms minus the sum of the first 3 terms; this is $[2(4)^2 + 4] - [2(3)^2 + 3] = 36 - 21 = 15$

6. (a) This is the probability that exactly one of the first 4 tosses is a head and the fifth is a tail; this gives $4(1/2)^4(1/2) = 1/8$.

7. (e) $y^2 = 4x^2 - 4ax + a^2 = x^2 - a$ gives the quadratic equation $3x^2 - 4ax + (a^2 + a) = 0$ and the discriminant is $4a^2 - 12a$ which equals 0 if $a = 3$.

8. (a) Tom's time is 1/10. Letting *r* be the unknown speed then John's total time is (1/2)/(9) + (1/2)/*r* . Equating the times gives 1/10 = 1/18 + 1/2*r* and solving for *r* gives the result.

9. (c) $\log_{10} 5^{20} = 20 (\log_{10}10 - \log_{10}2) \approx 20(1 - .301) \approx 14$ so $5^{20} \approx 10^{14}$

10. (e) They are respectively 2^{32}, 2^{27}, 2^{32}, 2^{64}, 2^{81}

11. (a) Let the persons be denoted *A*,*B*,*C*. The probability *B* has a different birth month than *A* is 11/12, and assuming this then the probability *C* has a different birth month than both *A* and *B is* 10/12; the answer is then (11/12)(10/12) = 55/72.

12. (d) The change in the east direction is $x = 20 \cos 30° + 20 \cos 60° = 10(1 + \sqrt{3})$. The change in the north direction is the same. The total distance is $(x^2 + x^2)^{1/2} =$

$[200(1 + \sqrt{3})^2]^{1/2}$.

13. (e) $= 3^3 (3^6 - 1) = 3^3\text{x}728 = 3^3\text{x}8\text{x}91 = 3^3\text{x}2^3\text{x}7\text{x}13$

14 (b) At the end *A* has (1/7)(6) = 6/7 ounces of acid and *B* has 6 – (1/7)(6) = 36/7 ounces of acid.

15. (d) Substitution of $x = -1$ *and* $x = 2$ gives $-b + (a + b) + 7 - 10 = 0$ and $8b + 4(a + b) - 14 - 10 = 0$ from which $a = 3$, $b = 1$. Division of $x^3 + 4x^2 - 7x - 10$ by $(x + 1)(x - 2) = x^2 - x - 2$ gives $x + 5$.

16. (d) If $a \le 4$ then $1/a + 1/b \ge 1/4$ and if $a > 8$ then $1/a + 1/b < 1/4$. The only cases are $a = 5$, $b = 20$; $a = 6$, $b = 12$; $a = b = 8$.

17. (b) There are a total of C(8,2) = (8x7)/2 = 28 combinations of 2 persons from 8. If there are 11 ties then there are 17 wins and the total number of points is 17x5 + 11x2.

18. (d) The vertices of the triangle are determined by the solving the equations pairwise simultaneously. They are $P = (-4,-4)$, $Q = (0,4)$ and $R = (2,2)$. Since the slopes of $y = x$ and $y = 4 - x$ are negative reciprocals, the angle at R is a right angle and the area is $(1/2)(PR)(QR) = (1/2)(6\sqrt{2})(2\sqrt{2}) = 12$.

19. (b) If n successively takes on the values 1,2,3,4,5,6,7,.. then 3^n mod 5 successively takes on the values 3,4,2,1,3,4,2,… , the sequence repeating after each 4 numbers ; this is seen from 3^{n+4} mod 5 = (3^nmod 5)(3^4 mod 5) = 3^n mod 5. Thus 3^{100} mod 5 = 3^4 mod 5.

20. (c) Using binomial approximations $\sqrt{17} - \sqrt{15} = (16 + 1)^{1/2} - (16 - 1)^{1/2} \approx [16^{1/2} + (1/2)(16)^{-1/2}] - [16^{1/2} + (1/2)(16)^{-1/2}(-1)] = 0.25$. Note the remaining terms in the approximations of each of $\sqrt{17}$ and $\sqrt{15}$ form decreasing series which alternate in sign and thus yield values in magnitude less than the first term which is less than 0.01.

21. (b) Let there be N women. For $n = 1,2,\ldots$ the nth woman knew $n + 10$ men so the Nth one knew $N + 10 = 50 - N$ men; solving gives $N = 20$.

22. (d) Solve $0 < x^2 - 3 < 1$ which is equivalent to $3 < x^2 < 4$.

23. (e) The distance from P to the center of C is 13 and this is the hypotenuse of a right triangle whose legs have length 5 and PQ; use the Pythagorean Theorem.

24. (b) $x(1) = 2^{1/2}$, $x(2) = 2^{3/4}$, $x(3) = 2^{7/8}$ and $x(4) = 2^{15/16}$.

25. (c) $x^2 + y^2 + z^2 = (x + y + z)^2 - 2(xy + xz + yz)$ and $1/x + 1/y + 1/z = (xy + xz + yz)/xyz$. Thus $x^2 + y^2 + z^2 = 20^2 - (2)(5)(10) = 300$

26. (a) The upper half portion of R consists of a sector of one of the circles subtended by 60° (this has an area of $\pi r^2/6$) and an additional small region whose area is $\pi r^2/6$ minus the area of an equilateral triangle with sides of length r (which has area $\sqrt{3}/4\, r^2$). Thus the total area inside both circles is $2(2\pi r^2/6 - \sqrt{3}/4\, r^2)$.

27. (d) If x is the solution and S the amount invested then $2S = S(1 + r)^{10}$ and $3S = S(1 + r)^x$. Then (using any base > 1 for the logarithm), $\log(1 + r) = (\log 2)/10 = (\log 3)/x$. Solve for x.

28. (b) Subtracting the second equation from the first gives $0 = (2 - c)x + 3$ and adding (–2) times the second equation to the third gives $0 = -cx + 9$. Solving simultaneously the resulting two equations gives $x = 3$ and $c = 3$. As a check, substituting in the three given equations gives $y = 5$ in each case.

29. (c) $= (1 - x^2)/x(1 - x^{1/2}) = (1 - x^2)(1 + x^{1/2})/x(1 - x^{1/2})(1 + x^{1/2}) = (1 - x)(1 + x)(1 + x^{1/2})/x(1 - x) = (1 + x)(1 + x^{1/2})/x$ is near $(2)(2)/1 = 4$ when x is near 1.

30. (e) If the first attempt has x soldiers on each side then $x^2 = S - 100$ and $(x + 1)^2 = S + 61$. Then $(x + 1)^2 = x^2 + 100 + 61$ which gives $x = 80$. Thus $S = 80^2 + 100 = 6500$.