Answers and Brief Solutions to E2003

1. (c) = (30x80 + 40x60 + 20x50)/(30 + 40 + 20) = 580/9 = 64 4/9

2. (a) x*x = 1/2 and x*(1/2) = x/(x +1/2) = 2x/(2x + 1)

3. (b) The discriminant b^2 - 4c must be a perfect square. The possible values for (b,c) are (2.1), (3,2), (4,3), (4,4),(5,4), (5,6), (6,5).

4. (d) Bill missed 20% of his free throws which was twice as many as the 10% that John missed.

5. (c) Each 5 years the value is multiplied by 1500/1000 = 3/2. Thus the answer is $1,000x$(3/2)^2$ = $2,250. Note: If $(1 + r/4)^{20} = 3/2$ then $(1 + r/4)^{40} = (3/2)^2$

6. (c) Method 1: There are $3^3 = 27$ possible selections for the three years. There are 3! = 6 orderings of the three schools. Thus the answer is 6/27 = 2/9. Method 2*:* The probability the second year gives a different school is 2/3; in this case the probability the third year is again different is 1/3. The answer is then (2/3)(1/3) = 2/9.

7. (e) The altitude divides the length of side 5 into two parts of length x (adjacent to the side of length 4) and 5 – x. Then there are two right triangles and by the Pythagorean Theorem $x^2 + h^2 = 4^2$ and $(5 – x)^2 + h^2 = 3^2$. Solving these gives x = 16/5 and h^2 = 16 – 256/25 = 144/25.

8. (b) n = 3 gives the only solution since for all such triples of numbers at least one is divisible by 3.

9. (a) Since 1,2,-1 are roots then P(x) = k(x – 1)(x – 2)(x + 1) for some k and -24 = k(-3)(-4)(-1) so k = 2 and P(3) = 2(2)(1)(4) = 16.

10. (e) For each number in {1,2,3,4,5} there are 2^4 = 16 subset of {1,2,3,4,5} containing that number. Thus the answer is 16(1 + 2 + 3 + 4 + 5) = 240.

11. (e) Let B_1,G_1 be the number of boys, girls in 1990 and B_2,G_2 the number of boys,girls in 2000. Then $B_2 = 1.1B_1$, $G_2 = 1.4G_1$ and $B_2 + G_2 = 1.3(B_1 + G_1)$. Combining gives $.1G_1 = .2B_1$.

12. (d) From the inequality $5x^2 + x – 3 > 2x^2 + 6x – 1$ obtain $3x^2 – 5x – 2 = (3x + 1)(x – 2) > 0$ which is true if x > 2 or x < -1/3.

13. (b) Since 616 = (2^3)(7)(11) it follows that n = (2)(7)(11) = 154.

14. (e) After evaporation the weight of the salt in the solution is still 10 pounds and the solution is 40% salt; thus the weight of the solution is 10/0.4 = 25 pounds.

15. (c) Squaring both sides gives $m^2 + 2n^2 + 2mn\sqrt{2} = 41 + 24\sqrt{2}$ giving $m^2 + 2n^2 = 41$ and 2mn = 24 since m,n are integers. These equations may be solved simultaneously to give m = 3 and n = 4.

16. (c) If a,b,c are the lengths of the sides then a + b + c = 16 and the sum of any two of a,b,c exceeds the third. The possible cases are the triples 7,7,2; 7,6,3; 7,5,4; 6,6,4; 6,5,5.

17. (c) From 2a + b = 6, 2d + e = 8; and a + 2b = 6, d + 2e = 4 it follows that a = b = 2 and e = 0, d = 4. Thus 2x + 2y = 8 and 4x = 12 which gives x = 3, y = 1.

18. (e) Let d be the distance between towns A and B and t be the time after they start until they meet. Then d = (d/6)t + (d/8)t; solve for t.

19. (d) Let x be the common value of the three logarithms; then $4^x = a$, $6^x = b$ and $9^x = a + b$. Hence $b/a = (3/2)^x = (a + b)/b$ from which $b^2 – ab – a^2 = 0$. Solve for b using the quadratic formula and form the ratio b/a.

20. (d) Let x be the unknown distance, T_1,T_2 the points of tangency and O_1,O_2 the centers of the respective circles C_1,C_2. Then the triangles PO_1T_1 and PO_2T_2 are similar giving the equality x/2 = (7 - x)/3.

21. (a) There are C(8,2)x10 = 280 games played. If the champion wins n games and the common difference of succesive wins is k then n + (n - k) + ... + (n - 7k) = 8n - 28k = 280 or 2n - 7k = 70. Then k must be even and the smallest possible value for k is 2 which gives 2n = 84.

22. (e) The line through (a,0) perpendicular to y = 2x has the equation y = -1/2 (x - a) and intersects the line y = 2x in the point (a/5,2a/5). The distance formula gives $2^2 = (4a/5)^2 + (2a/5)^2$ which simplifies to $100 = 20a^2$

23. (c) The equation of the line is y = -2x + 4. The equation of the radius drawn to this line is y = x/2 and the point of tangency is the point (8/5,4/5) which is the intersection of these two lines. From $r^2 = (8/5)^2 + (4/5)^2$ the solution results.

24. (a) $A^{24} = 2^{15}$, $B^{24} = 3^8$ and $C^{24} = 4^6 = 2^{12}$ .Since $2^7 > 3^4 > 2^6$ it follows that $A^{24} = 2^{15} > 2^7x2^7 > 3^4x3^4 = B^{24} > 2^6x2^6 = C^{24}$.

25. (b) The given fraction = $(2x\text{-}1)(2^x+1)/ 2(2^x-1) = (2^x+1)/2$ and if $0 < x < 0.1$ then $1 < 2^x < 2$ so the given fraction is between 1 and 3/2 and hence between 1 and 2.

26. (d) Let F be the midpoint of AB; then CF is perpendicular to AB. . Then tan 60° = CF/AF = CF/(3/2) gives CF =$3\sqrt{3}/2$ and therefore tan ∠CDE = CF/DF =CF/(1/2) gives the result.

27. (e) For some integer j, n = 72j + 41 = (18)(4)j + (18)(2) + 5 = 18(4j +2) + 5. Thus n – 5 is divisible by 18.

28. (a) If divided by 100 the remainders of 7^n for n = 0,1,2,3,4,5,6, are 1, 7, 49, 43, 1, 7, 49, …; thus every fourth remainder is 1. Also note $7^{100} = (7^4)^{25} = (24x100 + 1)^{25} = 100N + 1$ for some integer N using a binomial expansion since each term but the last is divisible by 100.

29. (b) Each vertex can be joined by a diagonal to 7 other vertices; this counts each diagonal twice. The answer is (10)(7)/2 = 35 diagonals

30. (d) n = 2m – 5 gives m – 3n = m – 3(2m – 5) = -5m + 15 = 5(3 – m)