Answers and Brief Solutions to E2003

1. (c) \( (30x80 + 40x60 + 20x50)/(30 + 40 + 20) = 580/9 = 64 \frac{4}{9} \)

2. (a) \( x^2 = \frac{1}{2} \) and \( x*(1/2) = x/(x + 1/2) = 2x/(2x + 1) \)

3. (b) The discriminant \( b^2 - 4c \) must be a perfect square. The possible values for \( (b,c) \) are (2,1), (3,2), (4,3), (4,4), (5,4), (5,6), (6,5).

4. (d) Bill missed 20\% of his free throws which was twice as many as the 10\% that John missed.

5. (c) Each 5 years the value is multiplied by \( 1500/1000 = 3/2 \). Thus the answer is \$1,000 \times (3/2)^2 = \$2,250 \). Note: If \( (1 + r/4)^{20} = 3/2 \) then \( (1 + r/4)^{40} = (3/2)^2 \)

6. (c) Method 1: There are \( 3^3 = 27 \) possible selections for the three years. There are \( 3! = 6 \) orderings of the three schools. Thus the answer is \( 6/27 = 2/9 \). Method 2: The probability the second year gives a different school is \( 2/3 \); in this case the probability the third year is again different is \( 1/3 \). The answer is then \( (2/3)(1/3) = 2/9 \).

7. (e) The altitude divides the length of side 5 into two parts of length \( x \) (adjacent to the side of length 4) and \( 5 - x \). Then there are two right triangles and by the Pythagorean Theorem \( x^2 + h^2 = 4^2 \) and \( (5-x)^2 + h^2 = 3^2 \). Solving these gives \( x = 16/5 \) and \( h^2 = 16 - 256/25 = 144/25 \).

8. (b) \( n = 3 \) gives the only solution since for all such triples of numbers at least one is divisible by 3.

9. (a) Since 1,2,-1 are roots then \( P(x) = k(x - 1)(x - 2)(x + 1) \) for some \( k \) and \(-24 = k(-3)(-4)(-1) \) so \( k = 2 \) and \( P(3) = 2(2)(1)(4) = 16 \).

10. (e) For each number in \{1,2,3,4,5\} there are \( 2^4 = 16 \) subset of \{1,2,3,4,5\} containing that number. Thus the answer is \( 16(1 + 2 + 3 + 4 + 5) = 240 \).

11. (e) Let \( B_1, G_1 \) be the number of boys, girls in 1990 and \( B_2, G_2 \) the number of boys, girls in 2000. Then \( B_2 = 1.1B_1, G_2 = 1.4G_1 \) and \( B_2 + G_2 = 1.3(B_1 + G_1) \). Combining gives \( .1G_1 = .2B_1 \).

12. (d) From the inequality \( 5x^2 + x - 3 > 2x^2 + 6x - 1 \) obtain \( 3x^2 - 5x - 2 = (3x + 1)(x - 2) > 0 \) which is true if \( x > 2 \) or \( x < -1/3 \).

13. (b) Since \( 616 = (2^3)(7)(11) \) it follows that \( n = (2)(7)(11) = 154 \).

14. (e) After evaporation the weight of the salt in the solution is still 10 pounds and the solution is 40\% salt; thus the weight of the solution is \( 10/0.4 = 25 \) pounds.
15. (c) Squaring both sides gives \( m^2 + 2n^2 + 2mn\sqrt{2} = 41 + 24\sqrt{2} \) giving \( m^2 + 2n^2 = 41 \) and \( 2mn = 24 \) since \( m,n \) are integers. These equations may be solved simultaneously to give \( m = 3 \) and \( n = 4 \).

16. (c) If \( a,b,c \) are the lengths of the sides then \( a + b + c = 16 \) and the sum of any two of \( a,b,c \) exceeds the third. The possible cases are the triples \( 7,7,2; 7,6,3; 7,5,4; 6,6,4; 6,5,5 \).

17. (c) From \( 2a + b = 6, \) \( 2d + e = 8; \) and \( a + 2b = 6, \) \( d + 2e = 4 \) it follows that \( a = b = 2 \) and \( e = 0, d = 4 \). Thus \( 2x + 2y = 8 \) and \( 4x = 12 \) which gives \( x = 3, y = 1 \).

18. (e) Let \( d \) be the distance between towns A and B and \( t \) be the time after they start until they meet. Then \( d = (d/6)t + (d/8)t; \) solve for \( t \).

19. (d) Let \( x \) be the common value of the three logarithms; then \( 4^x = a, \) \( 6^x = b \) and \( 9^x = a + b \). Hence \( b/a = (3/2)^x = (a + b)/b \) from which \( b^2 - ab - a^2 = 0 \). Solve for \( b \) using the quadratic formula and form the ratio \( b/a \).

20. (d) Let \( x \) be the unknown distance, \( T_1,T_2 \) the points of tangency and \( O_1,O_2 \) the centers of the respective circles \( C_1,C_2 \). Then the triangles \( PO_1T_1 \) and \( PO_2T_2 \) are similar giving the equality \( x/2 = (7 - x)/3 \).

21. (a) There are \( C(8,2)\times 10 = 280 \) games played. If the champion wins \( n \) games and the common difference of successive wins is \( k \) then \( n + (n - k) + \ldots + (n - 7k) = 8n - 28k = 280 \) or \( 2n - 7k = 70 \). Then \( k \) must be even and the smallest possible value for \( k \) is 2 which gives \( 2n = 84 \).

22. (e) The line through \( (a,0) \) perpendicular to \( y = 2x \) has the equation \( y = -1/2 \, (x - a) \) and intersects the line \( y = 2x \) in the point \( (a/5,2a/5) \). The distance formula gives \( 22 = (4a/5)^2 + (2a/5)^2 \) which simplifies to \( 100 = 20a^2 \).

23. (c) The equation of the line is \( y = -2x + 4 \). The equation of the radius drawn to this line is \( y = x/2 \) and the point of tangency is the point \( (8/5,4/5) \) which is the intersection of these two lines. From \( r^2 = (8/5)^2 + (4/5)^2 \) the solution results.

24. (a) \( A^{24} = 2^{15}, \) \( B^{24} = 3^8 \) and \( C^{24} = 4^6 \). Since \( 2^7 > 3^4 > 2^6 \) it follows that \( A^{24} = 2^{15} > 2^7x2^7 > 3^4x3^4 = B^{24} > 2^6x2^6 = C^{24} \).

25. (b) The given fraction = \( (2x-1)(2^k+1)/2(2^k-1) = (2^k+1)/2 \) and if \( 0 < x < 0.1 \) then \( 1 < 2^x < 2 \) so the given fraction is between 1 and 3/2 and hence between 1 and 2.

26. (d) Let \( F \) be the midpoint of \( AB; \) then \( CF \) is perpendicular to \( AB \). Then \( \tan 60° = CF/AF = CF/(3/2) \) gives \( CF = 3\sqrt{3}/2 \) and therefore \( \tan \angle CDE = CF/DF \) \( = CF/(1/2) \) gives the result.

27. (e) For some integer \( j, n = 72j + 41 = (18)(4j) + (18)(2) + 5 = 18(4j +2) + 5 \). Thus \( n - 5 \) is divisible by 18.
28. (a) If divided by 100 the remainders of \(7^n\) for \(n = 0, 1, 2, 3, 4, 5, 6\), are 1, 7, 49, 43, 1, 7, 49, \(\ldots\); thus every fourth remainder is 1. Also note \(7^{100} = (7^4)^{25} = (24\times100 + 1)^{25} = 100N + 1\) for some integer \(N\) using a binomial expansion since each term but the last is divisible by 100.

29. (b) Each vertex can be joined by a diagonal to 7 other vertices; this counts each diagonal twice. The answer is \((10)(7)/2 = 35\) diagonals.

30. (d) \(n = 2m – 5\) gives \(m – 3n = m – 3(2m – 5) = -5m + 15 = 5(3 – m)\)