ANSWERS AND BRIEF SOLUTIONS TO E2004

1 (a) Solve \((.75)(60) + N = (.8)(60 + N)\) for \(N\); this gives \(.2N = 3\) and hence \(N = 15\).

2: (d) Let \(x\) be the side of the triangle; then \(3x/4\) is the side of the square. The area of the square is \(9/16 x^2\), and of the triangle is \(\frac{1}{2} \text{ base} \times \text{ altitude} = \frac{1}{2} (x)(\sqrt{3} \times 2) = \sqrt{3} x^2/4\).

3 (b) By division \(3/7 = 0.428571\) (i.e. this is a repeating sequence of 6 digits). Since 2 is the second digit in the sequence and the remainder of the division of 50 by 6 is 2, the conclusion follows

4 (d) If \(\log_{10}x = y\) then \(\log_{10}100x = \log_{10}100 + \log_{10}x = 2 + y\)

5 (c) Let John be assigned a seat; then there are 5 seats remaining and 2 of them are next to him. Therefore the probability Mary is in one of those 2 seats is \(2/5\).

6: (e) It is the distance between the nearest points of intersection of the circles and the straight line \(y = x\) joining the centers (since this line is perpendicular to the parallel tangents at the intersection points). These points of intersection are \((2,2)\) and \((6,6)\) and the distance between them is \(4\sqrt{2}\)

7 (a) If there were \(b\) boys and \(g\) girls then \(3/4 b + 3/5 g = 2/3 (b + g)\). Thus \(1/12 b = 1/15 g\) or \(b/g = 4/5\).

8 (c) The terms is the set of numbers \(2 + 3d, d = 1,2,\ldots,19\) and the sum is \((2)(20) + (3)(19)(20)/2 = 610\); the average is \(610/20 = 30.5\).

9 (b) Adding the first and third equations give \(y = (a + 2)x\); adding the second and third gives \(z = 1 - 3x\). Substituting these into the third equation gives \((7 + a)x = 1\) which does not have a solution is \(a = -7\). The same result occurs when substituting into the other two equations. If \(a \neq -7\) then there is a solution for \(x,y,z\).

10 (e) Let the endpoints of \(C\) be \(A\) and \(B\). The area of the triangle with vertices \(A,B,O\) is \((1/2)(1/2)(\sqrt{3})\) and the area of the circular region bounded by the circle and segments \(OA\) and \(OB\) is \(\pi/3\) since the angle subtended by \(AB\) is \(120^\circ\)

11 (e) Substitute 1 for \(x\) and set the result equal to 0; this gives 4 - \(b = 0\).

12 (c) Squaring both sides and simplifying gives \(x^2 + 1/x^2 = 7\). Again squaring both sides and simplifying gives the result \(x^4 + 1/x^4 = 47\).

13 (d) If \(x\) is the unknown then \(4 + .7x = .5(10 + x)\); solving gives \(x = 5\).

14 (b) The prime factors of \(25!\) are the prime numbers less than or equal to 25: i.e. \(2,3,5,7,11,13,17,19\) and \(23\).
15 (a) If P is the amount invested and r is the rate then \(2P = P(1 + \frac{r}{4})^{40}\); solve for r.
(Note: The answer is very close to .07 = 7%) 

16 (d) Method 1: Let \(z = BC\). Then by the law of cosines \(y^2 = x^2 + z^2 - xz\). Solving for \(z\) using the quadratic formula gives the discriminant value \(4y^2 - 3x^2\); set this value equal to 0 and solve for \(y/x\).
Method 2: There is exactly one possible triangle if \(\angle ACB\) is a right angle and hence \(y/x = \sin 60^\circ\).

17 (d) In both cases they meet 2/3 of the distance from A to the position of II when A begins. If \(r\) is the rate of I and D is the distance between A and B then I goes 2D/3 if they start at the same time and \((2/3)(D - r/24)\) if I starts 5 minutes later. From \((2/3)D - (2/3)(D - r/24) = 2\) it follows that \(r = 72\).

18 (e) An equation of the line through the given points is \(y - 3 = \frac{5}{4} (x - 2)\). Thus \(x\) can assume any of the values \(2 + 4k, k = 0,1,2,\ldots, 7\)

19. (a) \(2^{10} < 2x10^3 < 3^7 < 3x10^3 < 5^5\) implies \(2^{100} < 3^{70} < 5^{50}\)

20 (b) From \(a = x^{1/10}\), \(b = x^{1/5}\) it follows that \(x = (abc)^2 = x^{1/5}x^{2/5}c^2\) and hence \(c = x^{1/5}\).

21 (b) For each \(i, i = 0,1,2,\ldots,10\) there are \((11 - i)\) tuples \((i,j,k)\) such that \(i + j + k = 10\); thus summing for all \(i = 0,1,2,\ldots,10\) there is a total of \(11 + 10 + 9 + \ldots + 1 = (11)(12)/2 = 66\).

22. (d) Let \(\theta\) (degrees) be the measure of the smaller interior angles of the rhombus; then the measure of the adjacent interior angles are 180 - \(\theta\). By the law of cosines \(4 = 2x^2 - 2x^2 \cos \theta\) and \(9 = 2x^2 + 2x^2 \cos \theta\); eliminating \(x\) gives \(\cos \theta = 5/13\). Substitute this value in one of the previous equations and solve for \(x\).

23 (d) If each National League team plays \(y\) games against American League teams then \(14x = 16y\) or \(7x = 8y\). The smallest positive integer value for \(x\) is 8

24. (b) There are \(9x8 = 72\) total ways to place X and O. For X and O to be in sub-squares with a common side: if X is in the middle square there are 4 ways for O; if X is in one of the 4 corners there are 2 ways for O; if X is in one of the 4 squares between corners there are 3 ways for O. Thus the answer is \((1x4 + 4x2 + 4x3)/72 = 1/3\).

25 (c) By calculation \(x_1 = \pi/(1 + \pi), x_2 = \pi/(1 + 2\pi)\) and if \(x_n = \pi/(1 + n\pi)\) then \(x_{n+1} = \pi/(1 + n\pi) \div [1 + \pi/(1 + n\pi)] = \pi/[1 + (n + 1)\pi]\)

26. (e) Since \(4n + 1\) is odd let \(4n + 1 = (2k + 1)^2\). Then \(n = k^2 + k\) so any positive integer substituted for \(k\) gives a value of \(n\) which makes \(4n + 1\) as a perfect square
27 (b) If \( x \leq -3 \) the equation is \(-(x - 5) - (x + 3) = a\) which has solution \( x = 1 - \frac{a}{2}\); if \(-3 < x < 5\) the equation is \(-(x - 5) + (x - 3) = a\) which has no solution for \( x \) since \( a > 10\); if \( x \geq 5\) the equation is \((x - 5) + (x + 3) = a\) which has solution \( x = 1 + \frac{a}{2}\). Thus the sum of solutions is \((1 - \frac{a}{2}) + (1 + \frac{a}{2}) = 2\),

28. (e) Completing squares gives \(3(x - y)^2 + (y - 2)^2 + 7\)

29. (a) The operations on the graph give the successive equations \(x = y^2 + 1\), \(x = y^2 + 3\), and \(x = (y - 2)^2 + 3\); substitute \(y = 4\) in the last equation.

30. (d) The statement is equivalent to \(x < 1\) and \(y \geq 7\); then only (d) is always true.