ANSWERS AND BRIEF SOLUTIONS TO E2004

1 (a) Solve (.75)(60) + N = (.8)(60 + N) for N; this gives .2N = 3 and hence N = 15.

2: (d) Let x be the side of the triangle; then 3x/4 is the side of the square. The area of the square is 9/16 x^2, and of the triangle is ½ base x altitude = ½ (x)($\sqrt{3}$ x/2) = $\sqrt{3}$ x^2/ 4.

3 (b) By division 3/7 = $0.\overline{428571}$ (i.e. this is a repeating sequence of 6 digits). Since 2 is the second digit in the sequence and the remainder of the division of 50 by 6 is 2, the conclusion follows

4 (d) If $\log_{10}x = y$ then $\log_{10}100x = \log_{10}100 + \log_{10}x = 2 + y$

5 (c) Let John be assigned a seat; then there are 5 seats remaining and 2 of them are next to him. Therefore the probability Mary is in one of those 2 seats is 2/5.

6: (e) It is the distance between the nearest points of intersection of the circles and the straight line y = x joining the centers (since this line is perpendicular to the parallel tangents at the intersection points). These points of intersection are (2,2) and (6,6) and the distance between them is $4\sqrt{2}$

7 (a) If there were b boys and g girls then 3/4 b + 3/5 g = 2/3 (b + g). Thus 1/12 b = 1/15 g or b/g = 4/5.

8 (c) The terms is the set of numbers 2 + 3d, d = 1,2,…,19 and the sum is (2)(20) + (3)(19)(20)/2 = 610; the average is 610/20 = 30.5.

9 (b) Adding the first and third equations give y = (a + 2)x; adding the second and third gives z = 1 – 3x. Substituting these into the third equation gives (7 + a)x = 1 which does not have a solution is a = -7. The same result occurs when substituting into the other two equations. If a $\neq$ -7 then there is a solution for x,y,z.

10 (e) Let the endpoints of C be A and B. The area of the triangle with vertices A,B,O is (1/2)(1/2)($\sqrt{3}$) and the area of the circular region bounded by the circle and segments OA and OB is π/3 since the angle subtended by AB is 120°

11 (e) Substitute 1 for x and set the result equal to 0; this gives 4 - b = 0.

12 (c) Squaring both sides and simplifying gives $x^2 + 1/x^2 = 7$. Again squaring both sides and simplifying gives the result $x^4 + 1/x^4 = 47$.

13 (d) If x is the unknown then 4 + .7x = .5(10 + x); solving gives x = 5.

14 (b) The prime factors of 25! are the prime numbers less than or equal to 25: i.e. 2,3,5,7,11,13,17,19 and 23.

15 (a) If P is the amount invested and r is the rate then $2P = P(1 + r/4)^{40}$; solve for r. (Note: The answer is very close to .07 = 7%)

16 (d) Method 1: Let z = BC. Then by the law of cosines $y^2 = x^2 + z^2 - xz$. Solving for z using the quadratic formula gives the discriminant value $4y^2 - 3x^2$; set this value equal to 0 and solve for y/x.
Method 2: There is exactly one possible triangle if $\angle ACB$ is a right angle and hence y/x = sin 60°.

17 (d) In both cases they meet 2/3 of the distance from A to the position of II when A begins. . If r is the rate of I and D is the distance between A and B then I goes 2D/3 if they start at the same time and (2/3)(D – r/24) if I starts 5 minutes later. From (2/3)D - (2/3)(D – r/24) = 2 it follows that r = 72.

18 (e) An equation of the line through the given points is y – 3 = 5/4 (x – 2). Thus x can assume any of the values 2 + 4k, k = 0,1,2,…, 7

19. (a) $2^{10} < 2x10^3 < 3^7 < 3x10^3 < 5^5$ implies $2^{100} < 3^{70} < 5^{50}$

20 (b) From $a = x^{1/10}$, $b = x^{1/5}$ it follows that $x = (abc)^2 = x^{1/5}x^{2/5}c^2$ and hence $c = x^{1/5.}$

21 (b) For each i, i = 0,1,2,…,10 there are (11 – i) tuples (i,j,k) such that i + j + k = 10; thus summing for all i = 0,1,2,…,10 there is a total of 11 + 10 + 9 + … + 1 = (11)(12)/2 = 66 .

22. (d) Let θ (degrees) be the measure of the smaller interior angles of the rhombus; then the measure of the adjacent interior angles are 180 - θ. By the law of cosines $4 = 2x^2 - 2x^2 \cos\theta$ and $9 = 2x^2 + 2x^2 \cos\theta$; eliminating x gives cos θ = 5/13. Substitute this value in one of the previous equations and solve for x.

23 (d) If each National League team plays y games against American League teams then 14x = 16y or 7x = 8y. The smallest positive integer value for x is 8

24. (b) There are 9x8 = 72 total ways to place X and O. For X and O to be in sub-squares with a common side: if X is in the middle square there are 4 ways for O; if X is in one of the 4 corners there are 2 ways for O; if X is in one of the 4 squares between corners there are 3 ways for O. Thus the answer is (1x4 + 4x2 + 4x3)/72 = 1/3.

25 (c) By calculation $x_1 = \pi/(1 + \pi)$, $x_2 = \pi/(1 + 2\pi)$ and if $x_n = \pi/(1 + n\pi)$ then $x_{n+1} = \pi/(1 + n\pi) \div [1 + \pi/(1 + n\pi)] = \pi/[1 + (n + 1)\pi]$

26. (e) Since 4n + 1 is odd let $4n + 1 = (2k + 1)^2$. Then $n = k^2 + k$ so any positive integer substituted for k gives a value of n which makes 4n + 1 as a perfect square

27 (b) If $x \leq -3$ the equation is $-(x - 5) - (x + 3) = a$ which has solution $x = 1 - a/2$; if $-3 < x < 5$ the equation is $-(x - 5) + (x - 3) = a$ which has no solution for x since $a > 10$; if $x \geq 5$ the equation is $(x - 5) + (x + 3) = a$ which has solution $x = 1 + a/2$ .Thus the sum of solutions is $(1 - a/2) + (1 + a/2) = 2$,

28. (e) Completing squares gives $3(x - y)^2 + (y - 2)^2 + 7$

29. (a) The operations on the graph give the successive equations $x = y^2 + 1$, $x = y^2 + 3$, and $x = (y - 2)^2 + 3$; substitute $y = 4$ in the last equation.

30. (d) The statement is equivalent to $x < 1$ and $y \geq 7$; then only (d) is always true..

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