Answers and Brief Solutions to E2005

1. (e) The total score was (30)(67.2) = 2016 and 2016/28 = 72.

2. (d) From $x(.8)(.8) = 49.92$ it follows that $x = 78$.

3. (a) $f(2x + 1) = A(2x + 1) + B = 2Ax + (A + B) = x = 1x + 0$ for all numbers x gives $2A = 1$ and $A + B = 0$ from which $A = 1/2$ and $B = - 1/2$.

4. (b) It is the area of the rectangle with vertices (0,0), (0,5), (6,5), (6,0) minus the sum of the areas of the three right triangles having vertices (0,0), (0,5), (3,5); (3,5), (6,5), (6,2); (0,0), (6,0), (6,2). This is 30 - 15/2 - 9/2 - 6.

5. (c) An equation of the line through $(3,a)$ and $(a,3)$ is $y - a = - (x - 3)$. Substituting $x = 1$, $y = 4$ gives $4 - a = -(-2)$.

6. (e) There are ten possible cases, all equally likely. Of these 7 yield a red ball and of these 7 cases 3 came from box 1. Thus the answer is 3/7.

7. (b) For each x there are $40 - x$ pairs. Thus there are $1 + 2 + 3 + \ldots + 39 = (39)(40)/2 = 780$ pairs.

8. (d) After the second transfer A has $(10 - x) + x[x/(10 + x)] = 8$. This simplifies to $8x = 20$.

9. (b) The sum of 4 successive integers must be an even number. Note 525 is divisible by 3,5,7 so in those cases use the quotient as the middle number. Note 525/6 = 87.5 so use the three integers immediately above and below 87.5

10. (a) If R is the number of right answers then $S = R - (N - R)/4$: solve for R

11 (a) From $200 = 100(1 + r)^{10}$ and $S + 200 = S(1 + r)^{20}$ it follows that $(1 + r)^{10} = 2$ and $S + 200 = S(2)^2$ from which $S = 200/3$.

12. (d) Let x be the height of the other pole, a the distance from the base of the 100 foot pole to the point on the ground below where the wires cross and b the distance between the bases of the two poles. Then by similar triangles $40/a = x/b$ and $40/(b - a) = 100/b$. From the second equation $b = 5/3\, a$; substitution in the first equation gives $x = 200/3$.

13. (c) Using the law of cosines, $4^2 = 3^2 + 2^2 - (2)(3)(2) \cos A$ which gives $\cos A = -1/4$; apply $\sin^2 A + \cos^2 A = 1$.

14: (d) All triangles have sides of length 6,8 and area $(1/2)(6)(8) \sin \theta = 24 \sin \theta$ where θ is the angle between the sides of length 6 and 8. This area is greatest when $\sin \theta = 1$ which implies $\theta = 90°$; by the Pythagorean Theorem the other side has length 10.

15 (e) Method I: If r,s are the other two roots then $r + s + 2 = -(-4)$ and $-2rs = 30$. Solving gives –3, 5 as values of r,s. Method II: $P(2) = 22 + 2A = 0$ gives $A = -11$. Division of $x^3 - 4x^2 + -11x + 30$ by $x - 2$ gives $x^2 - 2x - 15 = (x - 5)(x + 3)$ which has roots –3,5.

16 (b) By computation if $x_2 = a$ then $x_3 = 1 + a$, $x_4 = 1 + 2a$, $x_5 = 2 + 3a$, ... , $x_{10} = 21 + 34a$ and $21 + 34a = 100$ gives the result.

17. (d) Each time Jerry runs once around the track, Tom gains 2/9 of the track length. Thus after Jerry has run 9/2 times (and Tom 11/2 times) around the track, Tom has reached the same position as Jerry. Then Tom has run (11/2)(1/4) miles.

18 (c) 6 is a factor of each term of the sum except $1! = 1$ and $2! = 2$

19. (a) If $x < -5$ or $x > 2$ the product is positive and if $x = -5, -3, 0, 2$ the product is 0. Substitution of the remaining integers –4, -2, -1 and 1 gives negative values only for –4 and 1.

20. (e) $a = (2^4)^{15} = 16^{15}$; $b = (3^3)^{15} = 27^{15}$; $c = (5^2)^{15} = 25^{15}$

21. (c) From $y = 2/3\, x$ and $z = 1/3\, x$ and $x + y + z = 1$ it follows that $x = 1/2$, $y = 1/3$ and $z = 1/6$. Then $(1 - x)/(1 - y) = (1/2)/(2/3) = 3/4$.

22 (e) Let $x = a^{1/2}$ and $y = b^{1/3}$. Then $x + y = 1$ and $x^2 + y^2 = 5$. Thus $x^2 + (1 - x)^2 = 5$ from which $x = -1, 2$. The only real solution is $a = 2^2 = 4$

23 (d) By factoring the expression simplifies to $1 + 4/x$. If $|x - 1| < .001$ then $.9 < x < 1.1$ and $3 < 4/1.1 < 4/x < 4/.9 < 5$.

24. (c) $z = 1/2 - 1/2\, x$ and $y = 3/2 - 1/2\, z = 5/4 + 1/4\, x$. Then $ax - 2y + 5z = ax - 10/4 - 1/2\, x + 5/2 - 5/2\, x = 0$ gives $(a - 3)x = 0$ which is satisfied by every number x if $a = 3$. For each solution for x there are then solutions for y and z.

25. (d) Let x be the length of AD and $2\theta = \angle BAC$. Then $\cos\theta = 3/x$ and $3/5 = \cos 2\theta = 2\cos^2\theta - 1 = 18/x^2 - 1$ which gives $x = 3\sqrt{5}/2$.

26. (e) $1/\log 2 + 1/\log 4 = 1/\log 2 + 1/(2\log 2) = (3/2)/\log 2 = 1/\log c$ gives $c = 2^{2/3}$

27. (c) $5x + 51y = 5(100) + 51(1)$; hence $51(y - 1) = 5(100 - x)$. Since 5 and 51 are relatively prime it follows that $y - 1 = 5n$ and $100 - x = 51n$ for some positive integer n. Since $x = 100 - 51n > 0$ it follows that $n = 1$, giving $x = 49$ and $y = 6$.

28. (a) The probability A wins both the second and third games is (.8)(.8) = .64; the probability A loses the second and wins the third is (.2)(.2) = .04. The answer is the sum.

29 (b) Let the circle lie in the first quadrant and the tangent lines coincide with the x,y axes and intersect at the origin; let the point have coordinates (9,2). Then the center of the circle is (r,r) and the triangle with vertices (r,r), (9,2) and (r.2) is a right triangle with legs of length (r - 9) and (r – 2) and hypotenuse r. By the Pythagorean Theorem $r^2 = (r - 9)^2 + (r - 2)^2$ which simplifies to $(r - 5)(r - 17) = 0$.

30 (b) The implication ‘if $x < 7$ then $x < 4$’ is false only if ‘$x < 7$’ is true and ‘$x < 4$’ is false.