Answers and Brief Solutions to E2007

1. (b) $a = 3b/10$ so $3a/4b = 9/40$; $(9/40)\text{x}100 = 22.5$

2. (a) If x is the fourth term then $(1 + 7 + 8 + x)/4 = 7$ so $x = 12$.

3. (c) From $x = 2y - 12 = 2(2z - 6) - 12 = 4z - 24 = 4(2x - 8) - 24$ it follows that $7x = 56$

4. (e) Let y,z be the scores on the second and third tests. Then $y = 4x/3$ and $y = 5z/4$. Thus $z = (4/5)(4x/3)$.

5. (e) If $a = -2$, $b = -1$, $c = 1$ then (a), (b) and (d) are not true; if $a = 2$, $b = 1$, $c = -1$ then (c) is not true

6. (d) From $3x \geq y$ and $5y \geq z$ it follows that $15x \geq 5y \geq z$; hence from $y + z \geq 65$ it follows that $3x + 15x \geq 65$ which implies $x \geq 4$ since x is an integer. The values $x = 4$, $y = 12$, $z = 60$ satisfies the given conditions and shows that 4 is the minimum.

7. (e) $(\sin x + \cos x)^2 = 1 + 2 \sin x \cos x = 1 + \sin 2x = (5/4)^2$ gives $\sin 2x = 9/16$.

8. (c) Let $N = 11x + 9 = 7y + 5$; then $y = (11x + 4)/7$ and 6 is the smallest value of x for which y is an integer; thus $N = 75$.

9. (c) For $a = j$, $j = 1,2,\ldots,9$ there are $10 - j$ pairs. Thus the answer is $9 + 8 + 7 + \ldots + 1 = 45$

10. (a) The triangles APB and DPC are similar and thus the distance from P to DC is twice the distance from P to AB. Hence the answer is $(2/3)(6) = 4$.

11. (d) Every 100 years the value is increased by multiple of 2^5. Hence by the year 2001 the value has increased by a multiple of 2^{100}. Since $2^{10} = 1{,}024 > 1{,}000 = 10^3$ it follows that $2^{100} > 10^{30}$. Since $2^{10} < 2\text{x}10^3$ it follows that $2^{100} < (2\text{x}10^3)^{10} < 2\text{x}10^{33}$

12. (c) $7 = 2^x$ so the answer is $(2^x)^{3/x} = 2^3 = 8$

13. (b) Let r_b and r_t be the speeds of Bill and Tom respectively, and d the distance around the track. Then $60r_b + 60r_t = d = 50r_b + 80r_t$. Hence $r_t = 1/2\ r_b$ and $d = 50r_b + 40r_b = 90r_b$.

14. (d) The lines perpendicular to the two tangents have equations $y - 4 = 2(x - 3)$ and $y - 2 = -1/2\ (x - 5)$. They intersect at (1,0) which is the center of C and the distance from the center to either point of tangency is $2\sqrt{5}$

15. (a) After the transfer to *B*, *B* has 13 pounds of water and 2 pounds of salt. The final amount of water in *A* is $5 + 13y/15$ and the amount of salt in *A* is $2y/15$. Set $(2y/15)/(5 + y) = 1/20$ and solve for y.

16. (c) The two balls can be drawn in $A = (n + 5)(n + 4)/2$ ways and for $B = n(n - 1)/2$ of these ways both balls are green. Then $B/A \geq 1/2$ if $n^2 \geq 11n + 20$; the least n is $n = 13$.

17. (a) We consider all sequences of 0's and 1's of length 6 where the first digit is a 1 and the remaining 5 digits include at least four 0's. There are 5 sequences with four 0's and 1 with five 0's giving a total of 6.

18. (b) First replace x by $x - 1$, then y by $y - 2$ and finally x by $-x$ and y by $-y$; the resulting equation is $-y - 2 = (-x - 1)^2$ which simplifies to $y = -x^2 - 2x - 3$.

19. (c) Let $x = 100$. Then $A = (x + 1)/(2x + 5) = 1/2 - (3/2)/(2x + 5)$; $B = (2x + 3)/(4x + 9) = 1/2 - (3/2)/(4x + 9)$ and $C = (3x - 1)/(6x + 1) = 1/2 - (3/2)/(6x + 1)$. Thus $C > B > A$.

20. (a) If k is a positive integer then $p(k+1) - p(k) = 2k + 1 + b$. Thus $7 = 4 + 1 + b$ and hence $b = 2$. Therefore $p(4) - p(3) = 6 + 1 + 2 = 9$.

21. (b) If $x_2 = a$ then $x_3 = 1 + a$, $x_4 = 1 + 2a$, $x_5 = 2 + 3a, \ldots$, $x_{10} = 21 + 34a$ and $21 + 34a = 100$ gives the result.

22. (c) The possible winner sequences are *AA*, *ABA*, *BAA* which have respective probabilities 4/9, 4/27 and 4/27; adding gives the answer

23. (d) $B = (3,-1)$ and $C = (-1,3)$.

24. (d) Method 1: By sketching the graphs of $y = |x + 1|$ and $y = |2x - 1|$ it is seen there is only one solution if the graph of $y = |2x - 1|$ is shifted up 3/2 units so that the resulting graph lies above and intersects the graph of $y = |x + 1|$ at the single point (1/2,3/2).
Method 2: For $x \leq -1$ the given equation is $-x - 1 = -2x + 1 + c$ which has solution $x = 2 + c$ so there is no solution for $x \leq -1$ and $c > 0$. For $-1 < x \leq 1/2$ the equation $x + 1 = -2x + 1 + c$ has solution $x = c/3$. For $1/2 < x$ the equation $x + 1 = 2x - 1 + c$ has solution $x = 2 - c$; setting $c/3 = 2 - c$ gives $c = 3/2$.

25. (a) If $x \neq 2$ then $f(x) = -1/2x$ and if $1.9 < x < 2.1$ then $-1 < -1/3.8 < f(x) < -1/4.2 < 0$.

26. (b) By algebra, $y = 7x/(x - 7)$ and hence $x > 7$. Since $y \geq x$ it is also seen that $x \leq 14$; otherwise $1/x + 1/y < 1/7$. Solutions are $(x,y) = (8,56)$ and $(14,14)$. For other values of x, if $7 < x \leq 14$ then y is not an integer.

27. (d) Let G,B respectively be the number of senior girls, boys and T the total number of girls (boys) who attended the dance. Then $G/B = 2/3$ and $(T-B)/(T-G) = 3/7$. Dividing the numerators of these fractions by T and letting $g = G/T$ and $b = B/T$ gives $g/b = 2/3$ and $(1 – b)/(1 – g) = 3/7$. Solve the equations for g.

28. (d) Let d be the length of the altitude and θ the angle between the side of length

a and the hypotenuse. Then $\sin \theta = d/a$, $\cos \theta = d/b$ and hence $(d/a)^2 + (d/b)^2 = 1$. Solve for d and apply $c^2 = a^2 + b^2$

29. (c) This may be observed from the identity

$n^{100} + 1 = (n – 1)(n^{99} + n^{98} + n^{97} + \ldots + n + 1) + 2$. (Note: n can be any positive integer except 1)

30. (a) Let $x^3 + 5x^2 - 13x – 2 = (x^2 + ax + 1)(x – b) = x^3 + (a – b)x^2 + (1 – ab)x – b$. Then $a – b = 5$, $1 – ab = -13$ and $b = 2$ which is satisfied by $b = 2$, $a = 7$.