Answers and Brief Solutions to E2008

1. (c) From $36\pi - \pi r^2 = \pi r^2 - 4\pi$ it follows that $r^2 = 20$.

2. (a) Of the total group (1/3)x(3/5) = 1/5 passed the second time and (1/2)x (1-(2/3 + 1/5))= 1/15 passed the third time. Thus in the first three tries 2/3 + 1/5 + 1/15 = 14/15 passed the exam

3. (b) From $10^2 < 101 < 11^2$ and $31^2 < 999 < 32^2$ it follows that 31 – 10 = 21 is the answer.

4. (c) The new fare is $2.40 per mile and the old fare is $2.20 per mile. The increase is 20/220 and this is near 9%.

5. (d) If G is the number of girls and B of boys then the total weight of all students is $140(G + B) = 110G + 160B$ from which $30G = 20B$

6. (a) $\log_{10}125 = \log_{10}5^3 = 3 \log_{10}(10/2) = 3(\log_{10}10 - \log_{10} 2) = 3(1 - a)$

7 (b) The parabola must open upwards so $a > 0$. By completing the square it is seen that the x coordinate of the vertex is $-b/2a$ so $b < 0$. The parabola could cross the y axis either above or below the x axis so either of $c > 0$ or $c < 0$ is possible.

8. (e) Completing squares gives $3(x - y)^2 + (y - 2)^2 + 7$

9. (a) The new average minus the previous average is $(nx + y)/(n + 1) - x =$
$(y - x)/(n + 1)$

10. (b) $x^2 - x = 0$ implies $p(1) = 1 + a + b = 0$ and $p(0) = b = 0$. Thus $a + b = -1 + 0 = -1$

11. (e) $720 = 2^4 \text{x} 3^2 \text{x} 5$. Divisors of 720 are numbers of the form $(2^j)(3^k)(5^m)$ where j has one of the 5 values 0,1,2,3,4; k has one of the 3 values 0,1,2; m has one of the 2 values 0,1. The answer is 5x3x2 = 30.

12. (e) Method 1: The value multiplies by 3/2 each 4 years; thus after another 8 years the value is $(3/2)^3$($1,000) = $27,000/8
Method 2: $1500 = 1000(1 + r)^4$ so $(1 + r)^4 = 3/2$; thus $1000(1 + r)^{12} = 1000((1 + r)^4)^3 = 1000(3/2)^3$

13. (d) Let M be the number of married men (equals the number of married women). Then the total number of men is $7M/4$, the total number of women is $5M/3$ and the total number of married persons is $2M$. The answer is $2M/(7M/4 + 5M/3)$ which simplifies to 24/41

14. (d) If r is the average speed of the bicycle and t the time in hours for the trip then $50 = rt = (r + 5)(t – 1/2) = (r + 5)(50/r – 1/2) = 50 – r/2 + 250/r – 5/2$. This simplifies to $(r – 20)(r + 25) = 0$ which has solution $r = 20$.

15. (b) For $x = 1$ there are 18 values for y, for $x = 2$ there are 17 values for y; continuing gives the sum $18 + 17 + \ldots + 1 = (18)(19)/2 = 171$.

16. (b) If $x > 1/2$ then $x^n + 2x – 1 > 0$ so $0 < x < 1/2$. Thus $|x - 1/2| = |x^n/2| < (1/2)^3/2 < 0.1$

17. (d) There are two cases. If $x < -1/2$ then $x^2 – 2 = -4x – 2$ has solution $x = -4$. If $x > -1/2$ the equation $x - 2 = 4x + 2$ has solution $x = 2 + 2\sqrt{2}$

18. (e) Setting $2x = \pi/4 + 2\pi n$ or $2x = 3\pi/4 + 2\pi n$ for $n = 0,1$ gives the values $x = \pi/8$, $9\pi/8$ and $x = 3\pi/8$, $11\pi/8$; the sum is 3π.

19. (b) Let $v^5 = 3$, $w^6 = 4$, , $y^8 = 6$, $z^9 = 7$, $x^{12} = 10$. Then $v^{30} = 3^6 = 729 < 1024 = 4^5 = w^{30}$ shows $v < w$. Also $y^{24} = 6^3 = 216 < 256 = 4^4 = w^{24}$ shows $y < w$, $z^{18} = 7^2 = 49 < 64 = 4^3 = w^{18}$ shows $z < w$ and $x^{12}= 10 < 16 = 4^2 = w^{12}$ shows $x < w$.

20. (d) From $x^2 – y^2 = (x + y)(x – y) = 100 = (5^2)(2^2)$ the only possible pairs for $(x + y)$ and $(x – y)$ are (100,1), (50,2), (25,4), (20,5). The only case which gives integer values for x and y is $x + y = 50$ and $x – y = 2$ from which $x = 26$.

21. (c) Positive integers greater than 1 are relatively prime if they have no common factor. The only prime factors of 100 are 2 and 5. The prime numbers 3,7,11,13,17,19,23,29 are relatively prime to 100. Also $9 = 3^2$, $27 = 3^3$ and $21 = 3x7$ are relatively prime to 100.

22. (a) If r is the ratio then $2r^2 = 18$ gives $r = 3$, and $2r^4 = 162$ gives the sum 1+6+2 = 9.

23. (c) The three integers which are fixed can be selected in $C(6,3) = 20$ ways and for each of these ways the other three integers have 2 orderings in which none of those three are fixed.

24 (e) Triangle BPA is an isosceles 30°, 30°,120° triangle so $|BP| = 6$. Then in triangle BCP, $|BC| = 6 \cos 30° = 3\sqrt{3}$.

25. (c) $2^{10} = 1024 > 10^3$: then $2^{3000} = (2^{10})^{300} > (10^3)^{300} = 10^{900}$. Also $2^{10} < 2(10^3)$; then $2^{3000} < (2)^{300}(10^{900}) = (8^{100})(10^{900}) < (10^{100})(10^{900}) = 10^{1000}$. Note 10^n has n + 1 digits.

26. (a) In some order the areas of triangles with vertices *APB*, BPC. *APC* are $a/2$, $b/2$ and $c/2$. Thus a + b + c = 2x(area of the triangle *AB*C) = $2(\sqrt{3}/4) = \sqrt{3}/2$.

27. (c) Let *O* be the center of the circle, *M* the midpoint of the side *AB*, and *T* the point of tangency on the opposite side of the square. Then *O* is on the line *MT* since *OM* and *OT* are perpendicular to parallel sides of the square. Thus the right triangle *OMA* has hypotenuse *r* and sides 8 – *r* and 4. Therefore $r^2 = 4^2 + (8 - r)^2$ from which $r = 5$.

28. (d) **Method I.** From $.25(a + b + c + x + y) = a + x = b + y$ there results the equations $b + c - 3a = 3x - y$ and $a + c - 3b = 3y - x$. Adding the two equations gives $2c - 2a - 2b = 2(x + y)$.

Method II From $a + x = b + y = c/2$ it follows that $x + y + a + b = c$.

29. (b) The radius of *D* is *y* and applying the Pythagorean Theorem to the triangle with vertices (x,y), $(0,2)$, $(x,2)$ gives $(2 - y)^2 + x^2 = (y + 1)^2$. Solve for *y*.

30. (b) 5 tosses are needed if and only if the first 4 tosses produce 2 heads and 2 tails and this may occur in C(4,2) = 6 ways. The probability of this is $6/2^4 = 3/8$.