ANSWERS AND BRIEF SOLUTIONS TO ESSNER EXAM 28 2008-09

1. (b) The total score of the 5 exams was 5x70 = 350 and of the 4 exams, after dropping the lowest score, was 4x74 = 296. The difference is 350 – 296 = 54.

2. (a) The altitude divides the given triangle into two congruent isosceles right triangles, each of which has a hypotenuse of length 2, one side of length $\sqrt{2}$ and the altitude as the other side. Apply the Pythagorean to either of the congruent triangles.

3. (a) The difference is \$10,000(1.03^2 – 1.06).

4. (e) Let x be the distance from C to A and y the distance from B to D. Then $4 + y = 3x$ and $2(x + 4) = y$. Solving gives $x = 12$ and $y = 32$. Then d(C,D) = 12 + 4 + 32.

5. (d) Each team plays 4x3 = 12 games and there are 24 games in all. Team A wins 2 games and C wins 6 games, so D wins 24 – (2 + 7 + 6) = 9 games

6. (e) Let $N = 10x + y$. then $10x + y = 7x + 7y$, or $x = 2y$. Hence N has the form $N = 21y$ which gives the values 21, 42, 63, 84.

7. (d) After the two transfers the amount of water in A is $(8 – 4) + (10)[4/(x + 4)] = 6$. Solve for x.

8. (a) Let x be the length of the side of the square. Then $x^2 + (x/2)^2 = 1$ from which $x^2 = 4/5$.

9. (b) Let p be the fraction of the remaining votes needed by B. Then

$$.4x.6 + px.4 = .5 \text{ from which } p = .65.$$

10. (d) Let $a = 1/x$ and $b = 1/y$. Solve $2a + 3b = 21$ and $4a – b = 7$ to get $a = 3$ and $b = 5$. Therefore $x = 1/3$ and $y = 1/5$.

11. (b) From $n < 7^{120/80} = 7^{3/2} = \sqrt{343}$ and $18^2 = 324$, $19^2 = 361$ the answer follows

12. (d) Let the x coordinates of the three vertices be a,b,c. Then $(a + b)/2 = 1$; $(b + c)/2 = 3$ and $(a + c)/2 = 5$. Adding the three equations gives $a + b + c = 1 + 3 + 5 = 9$.

13. (c) The angle between the positive y axis and the line $y = mx$ is 30°. Thus the angle between the line $y = mx$ and the positive x axis is 60°. The answer is tan 60° = $\sqrt{3}$.

14. (e) The side of length 5 is the hypotenuse of a right triangle and subtends an angle of 180° on the circle; thus it is a diameter. The answer is then 5/2.

15. (b) Since $1/(b + 1/c) < 1$ then $a = 3$ and $(b + 1/c) = 5$. Since $1/c \le 1$ then $b = 4$ and $c = 1$.

16. (c) Let the sequence be a, ar, ar^2 ,ar^3 ,ar^4, ar^5, … .Then $ar^2 = 4$ and $(ar)(ar^3) = a^2r^4 = (ar^2)^2 = 16$

17. (d) By long division, $(n^2 – n)/(n – 3) = n + 2 + 6/(n – 3)$. It is easily verified that n – 3 divides 6 for n = 4,5,6 and 9. If n > 9 then n – 3 > 6 and hence n – 3 cannot divide 6.

18. (a) $3x = 3\log_2 5 = \log_2 125$ and $8^x = 2^{3x} = 125$ so $\log_5 8^x = \log_5 125 = 3$.

19. (b) In each case calculate the product of the average (middle) value and the number of terms in the range. Note that 80x**(**279/2) **=** 40x279. The sums are (a) 40x279 (b) 30x459 (c) 20x639 (d) 15x829 (e) 10x1,019. From 3x459>4x279 and 2x639 > 1,019 it follows that (c) > (a) and (c) > (e). From 2x459 > 829 it follows that (b) > (d). From 3x459 > 2x 639 it follows that (b) > (c).

20. (d) The succesive powers i^n of i starting with 1 are *i*, -1, -*i*, 1. From $i^{120} = (i^4)^{30} = 1$ it follows that $i^{123} = i^3 = -i$.

21. (b) Form a triangle with vertices from three consecutive vertices A,B,C of the hexagon. Then $\angle ABC = 120°$ and sides AB and BC each have length 1. Using the law of cosines, AC has length $\sqrt{3}$.

22. (c) From 1/x + 1/y = (x + y)/xy = a/xy = b it follows that y = a/bx. Then x + a/bx = a implies $bx^2 – abx + a = 0$ and the discriminant ab(ab – 4) of this quadratic equals 0 only if ab = 4.

23. (e) Let d miles be the distance between towns and t the hours if the train is on time. Then d = 50(t + 1/3) = 80(t – 1/6). Solving gives t = 1and hence d = 200/3 miles. Solving 60(1 + x) = 200/3 gives x = 1/9 hour = 20/3 minutes.

24. (d) If x < 1 the equation becomes $|-7| = 3$ and for x > 8 it becomes $|7| = 3$; neither of these cases gives a solution. For 1 < x < 8 there are the two equations (x – 1) + (x – 8) = 3 which has solution x = 6 and (x – 1) + (x – 8) = -3 which has solution x = 3.

25. (b) Let the ladder have length *L* and be positioned in the *xy* plane so that its bottom and top respectively have coordinates (Lcos θ,0) and (0,Lsin θ) where θ is the acute angle between the ladder and the positive x axis. If (x,y) is the midpoint of the ladder then x = (Lcos θ)/2 and y = (L sin θ)/2). It follows that $x^2 + y^2 = (L/2)^2$.

26. (e) The 6 selected numbers can be ordered in 6! ways; of these there are 3!x3! ways in which the three numbers assigned to the men are greater than the three assigned to the women. The answer is (3!x3!)/6!.

27. (c) Let x = 2 and y = 0. Then f(2,1) = f(f(2,0),0) = f(2,0) = 2

28. (d) Divide the equation $r^2 + r + 3 = 0$ by r^2 and rearrange terms to get $3(1/r)^2 + (1/r) + 1 = 0$. Thus (1/r) is a solution of $3x^2 + x + 1 = 0$; by symmetry (1/s) is also.

29. (a) $9! = 1x2^7x3^4x5x7$. From 3^4x5x7 there are the odd divisors of the form $3^mx5^nx7^p$ for five values 0,1,2,3,4 of m, two values 0,1 each of n and p. The answer is 5x2x2 = 20.

30. (e) Method I: Let *p* be the probability that Tom is the victor. If Tom wins the first roll then he is the victor. If Tom loses the first roll and wins the second roll then the game is returned to the original position. It follows that $p = 1/3 + (2/3)(1/3)p$ from which $p = 3/7$. Method II: Let W,L respectively denote win and loss for Tom. Then Tom wins with the sequences W, (LW)W, (LW)(LW)W, (LW)(LW)(LW)W,… . Addition of the corresponding probabilities gives a geometric series with initial term 1/3 and ratio (2/3)(1/3) = 2/9. The sum of the series is (1/3)/[1 – (2/9)] = 3/7.