ANSWERS AND BRIEF SOLUTIONS TO ESSNER EXAM 29 2009-10

1. (a) If x is the average on the next three rounds then $72 + 3x = 4(70) = 280$. Solve for x.

2. (a) Initially the mixture was 20% pecans. If the original amount of nuts in the can was x then after the removal of almonds the total amount in the can was .8x and the amount of pecans was .2x.

3. (c) If D is the distance between A and B, and r the average rate from B to A, then the total time for the round trip is $2D/50 = D/60 + D/r$. Solving gives $r = 300/7 = 42\ 6/7$.

4. (d) Starting with exponent 1, successive powers of 777 give as the last digit the sequence 7, 9, 3, 1, 7, 9, 3. (The number 7^7 would have the same last digit.)

5. (e) Since f(x) has degree 5 it can cross at most 5 times. Note f(2), f(6) and f(10) are all positive and f(1), f(5), and f(9) are all negative. Therefore the graph crosses the x axis between x values 1 and 2, 2 and 5, 5 and 6, 6 and 9, and 9 and 10.

6. (a) Divide each term by 3^{1000} to get $3^3 + 3^2$ divided by $3 – 1$.

7. (d) The prime factors of 210 are 2,3,5,7 and in each divisor of 210 each of the factors 2,3,5,7 occurs either 0 or 1 time; thus the answer is $2^4 = 16$. Alternatively sum the number of divisors with none, one, two, three and four factors to get respectively $1 + 4 + 6 + 4 + 1 = 16$.

8. (a) A second root is $x = 1 - \sqrt{2}\, i$. The product of $x – (1 + \sqrt{2}\, i)$ and $x – (1 - \sqrt{2}\, i)$ is $x^2 – 2x + 3$ and division of $x^4 – 4x^3 + 4x^2 - 9$ by $x^2 – 2x + 3$ gives $x^2 – 2x - 3$ which has roots -1,3.

9. (b) Let x red balls be added to the box. Then $(x + R)/(x + R + 30) = 3/5$. Solving for x gives $x = 45 – R$. Alternatively, after the addition the ratio of red balls to green balls will be 3 to 2; hence there will be $(3/2)(30) = 45$ red balls.

10. (a) Count the odd integers between 70 and 100 which are not divisible by 3,5, or 7. A number is divisible by 3 if the sum of the digits is divisible by 3 and by 5 if the unit’s digit is 0 or 5; only 77 and 91 are odd integers divisible by 7. This leaves six integers: 71, 73, 79, 83, 89,and 97.

11. (e) The roots are furthest apart when the discriminant is a maximum. The discriminant is $4c^2 - 4(2c^2 - 6c) = - 4c^2 + 24c = -4(c - 3)^2 + 36$ which is maximum when $c = 3$.

12. (c) Let r be the annual rate. By the compound interest formula $3 = (1 + r)^{15}$.
If $2 = (1 + r)^x$ then $2 = 3^{x/15}$. Solve $\log 2 = (x/15)\log 3$ for x.

13. (b) If A is one of the players then the number of possible selections of teammates of A is the number of possible choices $C(5,2) =10$ of 2 from the other 5 players.

14. (b) Let θ be the angle opposite the sides of length 2. Then the altitude to the third side is $2 \sin \theta$ and the length of the third side is $2(2 \cos \theta)$. Thus the area of the triangle is $1/2 \ (2 \sin \theta)(4 \cos \theta) = 4 \sin \theta \cos \theta = 2 \sin 2\theta$ which is a maximum when $\sin 2\theta = 1$.

15. (d) Let x be the length of the equal sides and y the length of the other side. Then $y + 2x = 100$, $0 < y < 2x$ and y is an even integer. This implies $26 \le x \le 49$ and y assumes even numbers between 2 and 48.

16. (b) The answer is $(3/4)(4/5) + (1/4)(1/5) = 13/20$

17. (c) The area of the parallelogram is $1 \text{x} 1 \text{x} \sin \theta$ and of the portion of the circle inside the parallelogram is $(\theta/2\pi) \text{x} \pi$.

18. (b) The line perpendicular to the tangent at (1,1) has equation $y - 1 = (x - 1)$, or $y = x$. Thus if (a,b) is the center then $b = a$. Equating the square of distances from (a,a) to (1,1) and to (3,2) gives $(1 - a)^2 + (1 - a)^2 = (3 - a)^2 + (2 - a)^2$ which yields $a = 11/6$.

19. (b) If a is the first term and d is the difference between each term and the next, then the sum of the first 10 terms is $10a + 45d$ and the 58^{th} term is $a + 57d$. Equating these gives, after simplification, $a = 4(d/3)$. Thus a must be a positive integer multiple of 4.

20. (e) Letting R,G denote the drawing of red and green balls there are the sequences RRR,RRG,RGR,GRR which have at least two red balls drawn. The total probability is $(2/3)^3 + 3(2/3)^2(1/3) = 20/27$.

21. (c) Let there be G gold, S silver and T total ribbons. Then $(T - G - S) = 3G$ and $.4T = G + S$. Substitution of $G = .4T - S$ into the first equation and simplifying gives $S = .2T$.

22. (b) The least common denominator of 13,39,65 is $13 \text{x} 3 \text{x} 5 = 195$. The inequalities may be written $45/195 < 5N/195 < 63/195$. Then $45 < 5N < 63$ implies $9 < N < 63/5$ which is true for N = 10,11,12.

23. (e) He made $nx/100$ of the first n shots and $(nx/100) + 1$ of the $n + 2$ shots. The answer is then $100[\{(nx/100) + 1\}/(n + 2)]$

24. (e) $N = (\sqrt{N})^2$ and hence $M = (\sqrt{N} + 1)^2 = N + 2\sqrt{N} + 1$

25. (c) Let x be the number of items sold. Then $D = S + C(x - N)$; solve for x.

26. (d) By factoring, $y = 1 + 4/x$. If $|x - 1| < .001$ then $.9 < x < 1.1$ and $3 < 4/1.1 < 4/x < 4/.9 < 5$. Add 1 to each term in the inequality.

27. (d) Since $\log_x 9 = 2\log_x 3$ the equation may be written $2y^2 – 5y + 2 = 0$ where $y = \log_x 3$. By the quadratic equation $y = 1/2$ or $y = 2$ and hence $x = 9$ or $x = \sqrt{3}$, which lies between 1 and 2

28. (a) <u>Method 1:</u> Substitute $z = 3x – 8$ into the first equation and factor to get $(x – 3)(y + 1) = 0$. Then $x = 3$ and from the other two equations $z = 1$ and $y = 2$.
<u>Method 2:</u> Solve the second and third equations for y and z in terms of x and substitute in the first equation to get $(x – 5)(x – 3) = 0$. Note the value $x = 5$ gives $y = -1$.

29. (b) To show B > A note that A = 999! = (999x1)x(998x2)x...x(501x499)x500 and for each parenthesis pair use the inequality $500^2 > (500 - k)(500 + k)$. To show A > C note that $[(999- k + 1)xk] \geq 999$ for k = 1 to 498 and $(501x499x500) > 999^2$ since 499 > 4.

30. (a) Let C1 and C2 be centered at the origin in the coordinate plane, and the chord be along the line y = c. Let w be the length of each of the three segments of the chord. Applying the Pythgorean Theorem to triangles with vertices (0,0), (0,c), (w/2,c) and (0,0), (0,c), (3w/2,c) gives $c^2 + (w/2)^2 = 16$ and $c^2 + (3w/2)^2 = 36$. From these $w = \sqrt{10}$.