# Answers and Brief Solutions to Essner Exam E2011 2010-2011

1. (e) Let F be the number of students that failed. Then 75(30 – F) + 60F = (30)(70). Simplifying gives 15F = 150.

2. (c) If a and b are the original prices of A and B then .94a and .9b are the marked down prices. Simplify .94a = .9b to obtain a/b = 45/47.

3. (a) Since the two roots are consecutive integers, $[a + (a^2 - 4b)^{1/2}]/2 = 1 + [a - (a^2 - 4b)^{1/2}]/2$. Algebraic simplification yields $a^2 = 4b + 1$.

4. (a) (2x + 1)/x = (4x)/(2x + 1) gives x = -1/4.

5 (d) log c = (log 2 )( log 8)/ (log 2 + log 8) = $3(\log 2)^2/4 \log 2$ = 3/4 log 2. Thus c = $2^{3/4}$

6. (e) Solve $x^2 = mx - 16$ using the quadratic formula. The discriminant is $m^2 - 64$ which equals 0 if m = 8.

7. (c) Let x,y,z be the lengths of the sides of the box where x < y < z. Then xy = 2, xz = 3 and yz = 4. The volume is xyz = $[(xy)(xz)(yz)]^{1/2} = 24^{1/2} = 2\sqrt{6}$ .

8. (a) Let A,B respectively denote the event that A,B wins, and let T denote the event of a tie. Then A wins exactly two games with each of the 6 sequences AAB, AAT, ABA, ATA, BAA, TAA each of which has probability $(1/2)^2(1/4) = 1/16$.

9. (a) In the isosceles triangle ΔADE, the angle ∠EAD has measure 60° + 90° = 150°. By the Law of Cosines if x is the length of DE then $x^2 = 1^2 + 2^2 + 2(2)(\sqrt{3}/2)$

10. (e) From xy = (x + 1)(y - 4) = (x - 3)(y + 2) it follows that -4x + y - 4 = 0 and 2x - 3y - 6 = 0. Solving gives x = -9/5, y = -16/5 which is not a possible value.

11. (c) For some number D, N = 60D + 41 = 12(5D) + 12(3) + 5 = 12(5D + 3) + 5

12. (c) Let $r_J$, $r_B$, and $r_T$ respectively denote the rates of John, Bill and Tom. Then $r_B/r_J$ = 9/10 and $r_T/r_B$ = 19/20. Hence $r_T/r_J$ = 171/200. Thus Tom finished 1 – 171/200 = 29/200 mile behind John.

13. (d) f(2x + 1) = A(2x + 1) + B = 2Ax + (A + B) = 6x + 8 for all numbers x gives 2A = 6 and A+ B = 8. It follows that A =3 and B = 5.

14. (e) Let M students take math and H students take history . Then M – 24 take math only and H – 24 take history only. From H = M/2 it follows that 150 = (M- 24) + (M/2 – 24) + 24 and this implies M = 116.

15. (c) $(1 + i)/(1 - i)x((1 + i)/(1 + i) = 2i/2 = i$. From $i^{10} = i^2 = -1$ the answer is $-(1 + i)$.

16. (e) Apply the quadratic formula to the equations $x^2 + 3x - 1 - a = 0$ and $x^2 + 3x - 1 + a = 0$. The discriminants are $9 + 4( a + 1)$ and $9 + 4(1 - a)$. Since $a > 0$, these are both greater than 9 provided $a < 1$.

17. (b) Let A be the number of gallons of acid added. Then $(A + 2)/(A + 10) = 1/4$ gives $A = 2/3$. Next let W be the number of gallons of water that is added. Then $(2/3 + 2)/(10 + 2/3 + W) = 1/5$ gives $W = 8/3$.

18. (d) From $200 = 100(1 + r)^{10}$ and $S + 200 = S(1 + r)^{20}$ it follows that $(1 + r)^{10} = 2$ and $S + 200 = S(2)^2$ from which $S = 200/3$

19. (d) The area of the square is $\pi$ and the sides of the square have length $\pi^{1/2}$. Let M be the midpoint of PQ and O the center of the circle and square. The length of OM is $\pi^{1/2}/2$. Apply the Pythagorean Theorem to $\Delta OMP$ to get $(x/2)^2 = 1 - \pi/4$. Solve for x.

20. (e) $\sqrt{100} = 10$; solve $|\sqrt{n} - 10| < 1$ to get $9 < \sqrt{n} < 11$ or $81 < n < 121$.

21. (a) $x_2 = 2^2 = 4$; $x_3 = 2^4 = 16$; $x_4 = 2^{16} = 2^6 * 2^{10} > 64{,}000$; $x_5 > 2^{64.000} = (2^{10})^{6,400} > (10^3)^{6,400} > 10^{10,000}$

22. (b) $P = (1 + 3,2) = (4,2)$; $Q = (-2,4)$; $(a,b) = (-2,-4)$. Note the 90° counter clock-wise rotation of $(x,y)$ is $(-y,x)$ and the reflection about the x axis of $(x,y)$ is $(x,-y)$.

23. (b) Let h be the probability of a head; then $1 - h$ is the probability of a tail and the probability the tosses have a different outcome is $2h(1 - h) = 2(h - h^2)$
$= 2[1/4 - (h - 1/2)^2] < 1/2$.

24. (d) For each $x = 1,2,\ldots,58,59$ there are $60 - x$ pairs. Thus there are $59 + 58 + 57 + \ldots + 2 + 1 = (59)(60)/2 = 1770$ pairs.

25. (e) It may be observed that $x = 1$ and $x = -1$ satisfy the equation. The equation may be written $x^4 - 5x^3 + 5x^2 + 5x - 6 = 0$. Dividing this by $x^2 - 1 = (x - 1)(x + 1)$ gives $x^2 - 5x + 6 = (x - 3)(x - 2)$. Thus the positive solutions are 1, 2, 3.

26. (a) The region consists of 4 regions, each of which has area equal to the difference of the area of a sector of a circle subtended by an angle of $\pi/4$ and the area of a right triangle which is 1/4. Thus the total area is $4(\pi/8 - 1/4)$

27. (a) Note $\pi/3 < \theta < \pi/2$. Therefore $B < A$ and from $A < 1$ and $C = B/A$ it follows that $B < C$. Also $A > \sin \pi/3 = \sqrt{3}/2$ and $C < \cot \pi/3 = 1/\sqrt{3}$; hence $C < A$.

28. (b) From $2x = -x$, $2x = x + 4$ and $-x = x+4$ the three vertices (0,0), (4,8) and (-2,2) of a triangle may be determined. The area may be found by subtracting the areas of two

right triangles from the area of the trapezoid with vertices (-2,0), (-2,2), (4,0) and (4,8). This gives (1/2)(8 + 2)(6) – (1/2)(2)(2) – (1/2)(4)(8) = 12.

29. (d) Let ABC denote the triangle with right angle at C, AC = 3 and CD = 2 where D is the endpoint of the altitude. Let $\theta = \angle ABC = \angle ACD$. Then $\cos\theta = 2/3$ and $\sin\theta = 2/y$ where y is the length of BC. Then $4/9 + 4/y^2 = 1$ implies $y^2 = 36/5$. By the Pythagorean Theorem $x^2 = 36/5 + 9 = 81/5$.

30. (b) The number of positive results is .96x1,000 + .01x600,000 = 6,960 and the number of positive results for those that have the disease is .96x1,000 = 960. Thus of those who have a positive result, the fraction who have the disease is 960/6,960 $\approx$ 1,000/7,000 = 1/7.