David Essner Mathematics Competition XXXI, 2011-2012

Preliminary Exam

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1. Tom, Alice and John took an exam. Alice scored 80. Tom scored 10 more than the average of the three, while John scored 16 less than the average of the three. The average of the three was then

(a) 72 (b) 74 (c) $75\frac{2}{3}$ (d) $76\frac{1}{3}$ (e) 78

Soln: **(b)** Let N be the average. Then $3N = 80 + (N + 10) + (N - 16)$ or $3N = 2N + 74$

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2. Two sides of an isosceles triangle have length 2 and 5. What is the area of the triangle?

(a) 5 (b) $2\sqrt{6}$ (c) $\sqrt{21}$ (d) $2\sqrt{5}$ (e) There is more than one possible value.

Soln: **(b)** The other side is 5. If h is the altitude to the side of length 2 then $h^2 = 5^2 - 1^2 = 24$. The area is then $(1/2)(2)(2\sqrt{6})$.

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3. What is the sum $1-3+5-7+9-\ldots+81$? (The terms increase in magnitude by 2 and alternate in sign.)

(a) -15 (b) 1 (c) 27 (d) 39 (e) 41

Soln: **(e)** $(1 - 3) + (5 - 7) + (9 - 11) + \ldots + (77 - 79) + 81 = (20)(-2) + 81$

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4. What is the area of a rectangle if the diagonals have length 1 and $60°$ is an angle of their intersection?

(a) $1/2$ (b) $\sqrt{2}/2$ (c) $\sqrt{3}/2$ (d) $\sqrt{3}/4$ (e) $(\sqrt{3}+1)/2$

Soln: **(d)** The rectangle is composed of two $30°, 60°$ right triangles each of which has hypotenuse 1 and sides of length $1/2$ and $\sqrt{3}/2$. The area of each triangle is then $\sqrt{3}/8$.

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5. A multiple choice test has 30 questions and 5 choices for each question. If a student answers all 30 questions and the score is [number right - (number wrong/4)] then which of the following is a possible score?

(a) -10 (b) 5.25 (c) 7.75 (d) 8.75 (e) 9.25

Soln: **(d)** If R is the number right then the score is $R - (30 - R)/4 = 5(R - 6)/4$. Thus the answer is a multiple of $5/4$. Note $8.75 = 7(5/4)$.

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6. Line L_1 has slope $1/2$ and Line L_2 has slope $1/3$. If L_1 and L_2 have the same y–intercept b and the sum of the x–intercepts of L_1 and L_2 is 10, then b equals

(a) $5/6$ (b) $-6/5$ (c) -2 (d) $3/2$ (e) 2

Soln: **(c)** The equations of L_1, L_2 are $y = x/2 + b$ and $y = x/3 + b$. Setting $y = 0$ gives the x intercepts as $-2b$ and $-3b$. From $-5b = 10$ it follows that $b = -2$.

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7. What is the value of $(\log_2 3)(\log_3 4)$?

(a) 3/4 (b) 4/3 (c) 3/2 (d) 2 (e) 8/3

Soln: **(d)** Let $x = \log_2 3$ and $y = \log_3 4$. Then $2^x = 3$ and $3^y = 4$. Hence $2^{xy} = 3^y = 4$ and $xy = 2$.

8. If $0 < x < \pi/2$ and $\sin x = 2\cos x$ then $(\sin x)(\cos x)$ equals

(a) 1/3 (b) 2/5 (c) 1/5 (d) 3/8 (e) $\sqrt{3}/4$

Soln: **(b)** $1 - \cos^2 x = \sin^2 x = 4\cos^2 x$ implies $5\cos^2 x = 1$ and hence $\cos^2 x = 1/5$ and $\sin^2 x = 4/5$. Thus $(\sin^2 x)(\cos^2 x) = 4/25$ and $(\sin x)(\cos x) = 2/5$.

9. How many positive integer pairs (m, n) satisfy the equation $2m + 7n = 835$?

(a) 44 (b) 51 (c) 60 (d) 71 (e) 119

Soln: **(c)** n can be any odd integer from 1 to 119 inclusive. There are $(1 + 119)/2 = 60$ such integers.

10. From a point P two tangent lines are drawn to a circle C. If A, B are the tangent points, O is the center of C, and $\angle APB = 30°$ then $\angle AOB$ equals

(a) 60° (b) 90° (c) 120° (d) 150° (e) 180°

Soln: **(d)** $\angle AOB = 360° - \angle APB - \angle PAO - \angle PBO = 360° - 30° - 90° - 90° = 150°$.

11. Let the function f satisfy $f(xy) = f(x)/y$ for all positive numbers x, y. If $f(5) = 10$ then what is the value of $f(8)$?

(a) 4 (b) 16/5 (c) 32/5 (d) 25/4 (e) There are many possible answers.

Soln: **(d)** $f(8) = f(5(8/5)) = 10/(8/5) = 25/4$

12. A full glass contains a mixture of 4/5 water and 1/5 alcohol. First 1/4 of the content is removed and replaced with alcohol. Then 1/3 of the resulting mixture is removed and replaced with alcohol. What fraction of the final mixture is alcohol?

(a) 13/20 (b) 11/20 (c) 2/5 (d) 3/5 (e) 47/60

Soln: **(d)** After the first step, the fraction of water is $(4/5)(3/4) = 3/5$. After the second step, the fraction of water is $(3/5)(2/3) = 2/5$. The final fraction of alcohol is then 1 - 2/5 = 3/5.

13. If a, b are real numbers, $a + b = 3$ and $a^2 + b^2 = 45$ then the value of $a^3 + b^3$ is

(a) 27 (b) 87 (c) 189 (d) 135 (e) not uniquely determined.

Soln: **(c)** $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$. From $9 - 45 = (a + b)^2 - (a^2 + b^2) = 2ab$ it follows that $ab = -18$. Thus $a^3 + b^3 = 3(45 + 18) = 189$.

14. Which of the following sets of three numbers form the lengths of the sides of an obtuse triangle?

(a) $5, 6, 12$ (b) $4, 6, 7$ (c) $4, 5, 6$ (d) $3, 4, 5$ (e) $2, 3, 4$

Soln: **(e)** An obtuse angle is opposite the longest side. From the law of cosines if the sides are a, b, c and c is the longest side then $c^2 - a^2 - b^2$ is positive if and only if the angle opposite c is obtuse. Note $4^2 > 2^2 + 3^2$. Also note the numbers in (a) are not the sides of a triangle.

15. In the Cartesian plane points P, Q are respectively in the first and fourth quadrants and each is a distance of 10 from the origin O. If segments OP, OQ respectively make an angle of $40°$, $20°$ with the positive x–axis and M is the midpoint of PQ, then the distance from O to M is

(a) $4\sqrt{2}$ (b) $5\sqrt{3}$ (c) 5 (d) $5\sqrt{2}$ (e) $4\sqrt{3}$

Soln: **(b)** The triangle OPQ is equilateral and triangles OPM and OQM are congruent $30°, 60°$ right triangles. The length of OM is $10 \sin 60°$.

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16. What is the real number value of x such that $2^x + 4^x = 12$?

(a) $\log_2 3$ (b) $\log_3 2$ (c) $\log_3 4$ (d) $\log_4 3$ (e) $\log_2 12$

Soln: **(a)** Let $y = 2^x$, then $y^2 = 4^x$. Hence $y^2 + y - 12 = (y + 4)(y - 3) = 0$ from which $y = -4, 3$. Only $y = 3$ gives a real number value for x, and $2^x = 3$ implies $x = \log_2 3$.

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17. The length of a chord of a circle is 14 and the shortest distance from the midpoint of the chord to the circle is 5. Then the radius of the circle equals

(a) 28/5 (b) 37/5 (c) 28/3 (d) 19/2 (e) $2\sqrt{6}$

Soln: **(b)** Let O be the center of the circle, A be the center of the chord and B an endpoint of the chord. Then applying the Pythagorean Theorem to the triangle AOB gives $(r-5)^2+7^2 = r^2$ from which $r = 37/5$.

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18. In a race A, B, and C run at a constant speed. When A finishes, B is 20 feet behind and C is 29 feet behind. When B finishes, C is 10 feet behind. What was the distance of the race?

(a) 100 feet (b) 150 feet (c) 200 feet (d) 250 feet (e) 300 feet

Soln: **(c)** Let d be the distance and r_A, r_B and r_C respectively be the speeds of A, B and C. Then $d/(d-29) = r_A/r_C = (r_A/r_B)(r_B/r_C) = [d/(d-20)][d/(d-10)]$ which simplifies to $d^2 - 29d = d^2 - 30d + 200$ and has solution $d = 200$.

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19. Three standard six-sided dice are rolled. What is the probability that the largest number that occurs is a five?

(a) 16/216 (b) 25/216 (c) 48/216 (d) 61/216 (e) 75/216

Soln: **(d)** There are $6^3 = 216$ possible outcomes. The numbers of outcomes with one, two and three 5s, and other numbers less than 5, are respectively $3(4^2)$, $3(4)$, 1 giving a total of 48 + 12 + 1 = 61. Thus the answer is 61/216.

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20. How many non-congruent rectangles are there which satisfy both (i) the lengths of the sides are integers and (ii) the perimeter equals the area?

(a) 0 (b) 1 (c) 2 (d) 3 (e) more than 3

Soln: **(c)** Let L and W be the integer valued lengths of the sides of a rectangle which has the same area and perimeter. Then $2L + 2W = LW$. Therefore $W = 2 + 4/(L - 2)$ and the only solutions with integer values are $L = 3, W = 6$ and $L = 4, W = 4$ and $L = 6, W = 3$. The cases $L = 3, W = 6$ and $L = 6, W = 3$ give congruent rectangles.

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21. If the polynomial $P(x) = x^3 - 4x^2 + ax + 30$, where a is a real number, has roots $2, r, s$, then what is $|r - s|$?

(a) 2 (b) 4 (c) 6 (d) 8 (e) There is more than one possible value.

Soln: **(d)** Since $P(x) = (x - 2)(x - r)(x - s)$, we have $r + s + 2 = -(-4)$ and $2rs = -30$. Solving gives -3, 5 as values of r, s.

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22. Mr. Jones invested the amount $\$P$. During the first year the value of the investment increased by $x\%$ and during the second year it decreased by $y\%$ of the amount at the end of the first year. If the value of the investment was exactly $\$P$ at the end of the second year then x equals

(a) y (b) $101y/100$ (c) $100y/101$ (d) $100y/(100-y)$ (e) $101y/(101-y)$

Soln: **(d)** Solve $P(1+x/100)(1-y/100) = P$ for x in terms of y.

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23. How many numbers between 1 and 1000 are integer multiples of either 6 or 15?

(a) 265 (b) 232 (c) 221 (d) 199 (e) 33

Soln: **(d)** Every sixth number is a multiple of 6. Thus from 1 to 1000 there are 166 multiples of 6. Similarly, from 1 to 1000 there are 66 multiples of 15, but half of these are also multiples of 6. Therefore the answer is $166 + 33 = 199$.

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24. Three men and three women are assigned different numbers selected at random from the integers 1 through 9. What is the probability the three numbers assigned to the men are all greater than the three numbers assigned to the women?

(a) 1/2 (b) 1/6 (c) 1/9 (d) 1/20 (e) 1/120

Soln: **(d)**. There are $6! = 720$ orderings of the six assigned numbers and for $3! \times 3! = 36$ of these orderings the numbers assigned to the men are all greater than the numbers assigned to the women. Thus the answer is $36/720 = 1/20$.

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25. If $y = x^2 - 2x$, what is the sum of all distinct real numbers x such that $(y+2)^y = 1$?

(a) -1 (b) 1 (c) 2 (d) 3 (e) 3/2

Soln: **(d)** $(y+2)^y = 1$ if $y = 0$ or if $y + 2 = 1$. Solving $x^2 - 2x = 0$ gives $x = 0, 2$ and solving $x^2 - 2x = -1$ gives $x = 1$. (Note: $y + 2 = -1$ does not give a solution for x).

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26. The difference $\sqrt{10} - 3$ is nearest to which of the following fractions?

(a) 1/3 (b) 3/10 (c) 2/9 (d) 1/4 (e) 1/6

Soln: **(e)** Method 1: Use the binomial expansion $(9+1)^{1/2} = 9^{1/2} + (1/2)9^{-1/2}(1) + [(1/2)(-1/2)9^{-3/2}]/2! + \ldots = 3 + 1/6 - 1/216 + \cdots$. The sum of the terms after 1/6 is very small compared to 1/6.
Method 2: $\sqrt{10} - 3 = 1/(\sqrt{10} + 3)$ and $6 < \sqrt{10} + 3 < 3.2 + 3 = 6.2$

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27. In the Cartesian plane if a line through the point $(10, 0)$ is tangent to the circle $x^2 + y^2 = 25$ at the point (a, b) then a equals

(a) $5/2$ (b) $5/\sqrt{2}$ (c) $5/\sqrt{3}$ (d) $5\sqrt{3}/2$ (e) $12/5$

Soln: **(a)** The slope of the line from $(0, 0)$ to (a, b) is the negative reciprocal of the slope of the tangent line. Solve simultaneously $b/a = (10-a)/b$ and $a^2 + b^2 = 25$ to get $10a = 25$.

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28. If $50! = N(10^k)$, where N is not a multiple of 10, then k equals

(a) 5 (b) 7 (c) 12 (d) 15 (e) 20

Soln: **(c)** The number 5 appears 12 times in the prime factorization of 50!. Since 5 is a prime factor of 10, all multiples of 10 must have a factor of 5.

29. At a high school 2/5 of the students are boys and 1/3 of the seniors are boys. If 1/5 of the boys are seniors then what fraction of the girls are seniors?

(a) 7/25 (b) 3/10 (c) 6/25 (d) 4/15 (e) 1/4

Soln: **(d)** Let B, G, B_S, G_S respectively denote the number of boys, girls, senior boys and senior girls. Then $(2/5)(B+G) = B$, $(1/3)(B_S + G_S) = B_S$ and $B/5 = B_S$. Thus $G = (3/2)B$ and $G_S = 2B_S$. Hence $G_S/G = 2B_S/(3/2)B = (4/3)(B_S/B) = 4/15$.

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30. The ratio $\dfrac{(1+i)^{10}}{(1-i)^7}$, where $i^2 = -1$, equals

(a) $4 - i$ (b) $2 + 2i$ (c) $1 - 2i$ (d) $3 - 3i$ (e) $4i - 1$

Soln: **(b)** $(1+i)/(1-i) = (1+i)(1+i)/(1-i)(1+i) = 2i/2 = i$. Also $(1+i)^2 = 2i$ and $(1+i)^3 = -2+2i$. Thus $(1+i)^{10}/(1-i)^7 = (1+i)^3[(1+i)/(1-i)]^7 = (-2+2i)(-i) = 2+2i$.