David Essner Mathematics Competition XXXII, 2012-2013

Preliminary Exam

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1. How many ways are there of choosing a, b from $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ such that $a < b$ and $a+b$ is a multiple of 3?

(a) 6 (b) 9 (c) 12 (d) 18 (e) 36

Soln: **(c)** 3×3 from $\{1, 4, 7\}$ and $\{2, 5, 8\}$, then another 3 from $\{3, 6, 9\}$

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2. John has taken a total of n tests each worth 100 points, and his average was 78. If on the last test he had made 91 instead of 70, then his average would have been 81. What is n?

(a) 7 (b) 6 (c) 5 (d) 4 (e) 3

Soln: **(a)** $78n - 70 + 91 = 81n$ so $21 = 3n$.

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3. Pat sold two DVDs on eBay for \$24 each. If she made a profit of 20% on one and a loss of 20% on the other, then what was the net result of the sale of the two items?

(a) \$1 profit (b) \$1 loss (c) \$2 profit (d) \$2 loss (e) no profit or loss

Soln: **(d)** ...

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4. If $\dfrac{x}{y-6} = \dfrac{y}{z-3} = \dfrac{z}{x-4} = 2$ then x equals

(a) 12 (b) 11 (c) 10 (d) 9 (e) 8

Soln: **(e)** Straightforward algebra...

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5. Jar A has 2 gallons and Jar B has 3 gallons of acid solution. If the fraction of acid in Jar A is 2/5 and the fraction of acid in the combined contents of Jar A and Jar B would be 1/2, then what is the fraction of acid in Jar B?

(a) 8/15 (b) 3/5 (c) 7/10 (d) 14/25 (e) 17/30

Soln: **(e)** $\frac{2}{5}2 + x3 = \frac{1}{2}5$ so $3x = \frac{5}{2} - \frac{4}{5} = \frac{17}{10}$.

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6. For how many integers x from 1 to 100 is $x^2 + x$ a multiple of 7?

(a) 21 (b) 25 (c) 28 (d) 32 (e) 36

Soln: **(c)** $7|x(x+1)$ implies $7|x$ or $7|x+1$. Then note $14 < 100/7 < 15$.

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7. In how many ways can 8 persons be grouped into 4 pairs?

(a) 128 (b) 105 (c) 96 (d) 72 (e) 64

Soln: **(b)** $7 \cdot 5 \cdot 3 = 105$

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8. Given a triangle ΔABC, if sides AB and AC each have length 3 and BC has length 2 then the length of the altitude from B to side AC is

(a) 3/2 (b) $\sqrt{3}$ (c) $4\sqrt{2}/3$ (d) $2\sqrt{3}/3$ (e) 7/4

Soln: **(c)** Let y be the altitude and x be the leg from the altitude to A. Then $y^2 = 3^2 - x^2 = 2^2 - (3-x)^2$ so $x = 7/3$ and $y^2 = 32/9$.

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9. If a, b are positive integers, $a \leq b \leq 9$, then how many different triangles are there with sides of length a, b, and 9?

(a) 25 (b) 22 (c) 32 (d) 29 (e) 45

Soln: **(a)** Use $a + b < 9$ with $a \leq b$ and count.

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10. The graphs of $y = x - 3$ and $y = mx + 1$ meet at a point whose x and y coordinates are each positive if and only if m is in the interval

(a) $(-1, 3)$ (b) $(1/3, 1)$ (c) $(1/3, 3)$ (d) $(-1/3, 1)$ (e) $(-1, 1/3)$

Soln: **(d)** When $m = -1/3$ the lines intersect on the x–axis. When $m = 1$ the lines are parallel.

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11. If $|x - 1| < .1$ and $x \neq 1$ then $(x^2 + 3x - 4)/(x^2 - x)$ must satisfy

(a) $x < 0$ (b) $0 \leq x < 2$ (c) $2 \leq x < 4$ (d) $4 \leq x < 6$ (e) $6 \leq x$

Soln: **(d)** Since $x \neq 1$, $(x^2 + 3x - 4)/(x^2 - x) = 1 + 4/x$. Near $x = 1$, this is near 5.

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12. Two dice are rolled; the value on each die is a random integer 1 through 6. If x is the value on one die and y is the value on the other die, then what is the probability that $|x - y|$ is less than 2?

(a) 1/3 (b) 1/2 (c) 7/18 (d) 5/12 (e) 4/9

Soln: **(e)** Count: out of 36, $|x - y| = 1$ for $2 \cdot 5$ of them and $x = y$ for 6 of them. $16/36 = 4/9$.

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13. Which number is the largest?

(a) 2^{48} (b) 3^{42} (c) 4^{30} (d) 6^{24} (e) 9^{18}

Soln: **(b)** Use that $3^2 > 2^3$...

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14. What is the largest prime factor of $2^{16} - 2^{13} + 2^8$?

(a) 5 (b) 7 (c) 11 (d) 13 (e) 17

Soln: **(a)** $2^{16} - 2^{13} + 2^8 = 2^8(2^8 - 2^5 + 1) = 2^8(2^4 - 1)^2 = 2^8(3 \cdot 5)^2$.

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15. If a, b are positive integers and $(a + 2b)(a - b) = 27$, then $2a + b$ is:

(a) 6 (b) 9 (c) 12 (d) 15 (e) not uniquely determined

Soln: **(c)** $27 = 1 \cdot 27 = 3 \cdot 9$ so either $a + 2b = 27$ or 9. Conclude $a = 5, b = 2$.

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16. If ΔABC is a right triangle with side AB having length 3 and hypotenuse AC having length 5, then what is the length of the angle bisector AD of $\angle BAC$, where D is on the side BC?

(a) 4 (b) $\sqrt{13}$ (c) $4\sqrt{5}/3$ (d) $3\sqrt{5}/2$ (e) $5\sqrt{3}/2$

Soln: **(d)** $\cos\theta = 3/5$ and $\cos^2\theta/2 = (1 + \cos\theta)/2 = (1 + 3/5)/2 = 4/5$. $AD = 3/\cos(\theta/2) = 3\sqrt{5}/2$.

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17. A box contains 4 balls, one each of the colors light blue, dark blue, light red, and dark red. A ball is drawn at random, returned to the box, and then a second ball is drawn at random. If at least one of the drawn balls was light blue, what is the probability that neither of the drawn balls was a shade of red?

(a) 3/8 (b) 3/7 (c) 2/7 (d) 1/3 (e) 1/4

Soln: **(b)** Standard probability calculation...

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18. If $12x = 0.12121212...$ (the digits 12 repeat indefinitely) then x equals

(a) 1/99 (b) 1/101 (c) 0.0099 (d) 0.011 (e) 0.012

Soln: **(a)** $100(12x) - 12x = 12$ so $x = 1/99$.

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19. Let N be the smallest positive integer such that if N is divided by 7 the remainder is 3 and if N is divided by 11 the remainder is 4. The sum of the digits of N is

(a) 7 (b) 14 (c) 11 (d) 9 (e) 12

Soln: **(b)** $N = 7x + 3 = 11y + 4$ for some x, y. Can just check that $N = 59$ is the smallest positive such.

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20. If the polynomial $x^2 - 4x + 2$ has roots r and s and the polynomial $x^2 + bx + c$ has roots $r + 1$ and $s + 1$, then $b + c$ equals

(a) 1 (b) 2 (c) 3 (d) 5 (e) 7

Soln: **(a)** $r, s = 2 \pm \sqrt{2}$ so $r + 1, s + 1 = 3 \pm \sqrt{2}$. Since $-b$ is their sum and c is their product, we get $b + c = -6 + 7 = 1$.

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21. In the binomial expansion of $(x^2 - y^3)^8$ what is the coefficient of the term $x^a y^b$ with $a - b = 1$?

(a) 28 (b) -28 (c) 56 (d) -56 (e) 0

Soln: **(d)** Since $2n - 3(8 - n) = 1$ only for $n = 5$, the first power of x only appears as the 5th term of the expansion. Hence the coefficient is $(-1)^5 \frac{8!}{5!3!} = -56$.

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22. In the Cartesian plane, if a line through the point $(10, 0)$ is tangent to the circle $x^2 + y^2 = 25$ at the point (a, b) then a is

(a) $5/2$ (b) $5/\sqrt{2}$ (c) $5\sqrt{2}/4$ (d) $5\sqrt{3}/2$ (e) not uniquely determined

Soln: **(a)** We have a right triangle in which the radius of the circle to (a, b) and a segment of the tangent line are legs. Since the ratio hypotenuse:radius = 2:1, the radius is at an angle $\theta = \pi/3$. Hence $a = 5\cos\pi/3 = 5/2$.

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23. In the Cartesian plane, what is the radius of the circle that passes through the points $(0, 0)$, $(4, 0)$, and $(0, -2)$?

(a) 3 (b) $7/3$ (c) $5/2$ (d) $\sqrt{7}$ (e) $\sqrt{5}$

Soln: **(e)** The three points are corners of a rectangle; the circle must have the same center as this rectangle: $(2, -1)$. The distance from $(0, 0)$ to $(2, -1)$ is $\sqrt{5}$.

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24. If m, n are positive integers and $n! = 210m!$ then $m + n$ equals

(a) 8 (b) 11 (c) 12 (d) 15 (e) 21

Soln: **(b)** $210 = 7 \cdot 6 \cdot 5$ so $n = 7$ and $m = 4$.

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25. For any real number x, let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x. If $a^2\lfloor a \rfloor = 25/2$ and $b\lfloor b \rfloor = 10$ then $b - a$ equals

(a) $7/2$ (b) $11/4$ (c) $11/2$ (d) $7/8$ (e) $5/6$

Soln: **(e)** $a = 2 + \frac{1}{2}$, $b = 3 + \frac{1}{3}$.

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26. The two equations $|y - x| = 1$ and $y/x = xy$ have how many simultaneous solution pairs (x, y)?

(a) 6 (b) 4 (c) 3 (d) 2 (e) 1

Soln: **(b)** $x = y \pm 1$ and either $y = 0$ or $x^2 = 1$.

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27. A sequence of 9 terms has the property that, except for the first two, each term is the sum of all the preceding terms in the sequence. If the 9th term is 512, then the sum of the first two terms is

(a) 4 (b) 6 (c) 8 (d) 12 (e) 16

Soln: **(c)**. The sequence is $a, b, a + b, 2(a + b), 4(a + b), 8(a + b), 16(a + b), 32(a + b), 64(a + b)$.

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28. Let $f(x)$ denote a real valued function. If $f(1 + f(x)) = 2f(x)$ and $f(0) = -2$ then $f(-3)$ equals

(a) -8 (b) -6 (c) -4 (d) -1 (e) 3

Soln: **(a)** $f(1 + f(0)) = 2f(0)$ so $f(-1) = -4$. Then $f(1 + f(-1)) = 2f(-1)$ so $f(-3) = -8$.

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29. A circle of radius 1 and a square have the same center and equal areas. If the circle intersects one side of the square in points P and Q, then what is the length of the segment PQ?

(a) $\pi/2$ (b) $\pi - 3/2$ (c) $(\pi - 3)/2$ (d) $\sqrt{4 - \pi}$ (e) $\frac{1}{2}\sqrt{1 + \pi}$

Soln: **(d)** There is an isosceles triangle with sides PQ and two radii. The altitude to PQ has length $\sqrt{\pi}/2$. So the length of PQ squared is $4(1^2 - \pi/4) = 4 - \pi$.

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30. If $\log_2 x^4 + \log_x 2 = 4$ then x equals

(a) $1/4$ (b) $\sqrt{2}$ (c) 2 (d) $1/2$ (e) $2\sqrt{2}$

Soln: **(b)** Just check: $\sqrt{2}^4 = 4$ and $\log_2 4 = 2$. Also $\log_{\sqrt{2}} 2 = 2$.
