# Optimal placement of Marine Protected Areas

Patrick De Leenheer, Mathematics, UF

*Everything Disperses to Miami, December 2012*

# Marine Protected Areas

- Restrict or forbid fishing to:
  - Protect over-harvested stocks
  - Restore habitats
  - Protect species diversity
- Effect on fishing yields still debated

# Motivation

1. When to install MPA?
2. Where to install it?

We need a model for the fish dynamics, and a reasonable measure to be maximized.

$\rightarrow$ use of *optimal control theory*.

# First paper taking optimal control perspective

*M.G. Neubert, Ecology Letters 6, 843-849 (2003)*

$U_T = DU_{XX} + rU(1 - U/K) - H(X)U, \;\; -L/2 < X < L/2$

$\mathbf{BC}: U(-L/2, T) = U(L/2, T) = 0$ for all $T \geq 0$

- $DU_{XX}$: diffusion
- $rU(1 - U/K)$: logistic growth
- $H(X)U$: fishing rate

Find $H(X)$ in $[0, \bar{H}]$ that maximizes steady state yield:

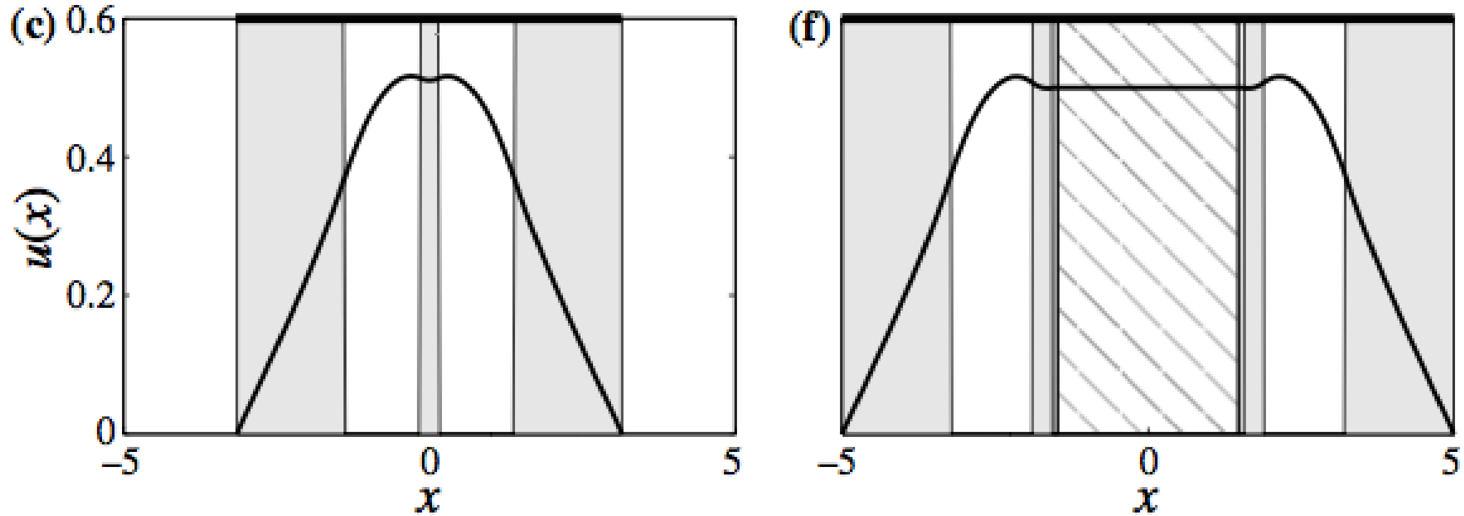
$$J(H(X)) = \frac{1}{L} \int_{-L/2}^{L/2} H(X)U(X)dX$$

# First paper taking optimal control perspective II

*M.G. Neubert, Ecology Letters 6, 843-849 (2003)*

- Intriguing numerical results.
- Cases of singular optimal control
- $L$ large $\Longrightarrow$ network of MPA's.
- Implementation problems: accumulation points in switches in control law.

# First paper taking optimal control perspective III




(c) $l = 6.3 \rightarrow 2$ reserves. (f) $l = 10 \rightarrow \infty$ many reserves.

$h^* = 0$: white; no fishing.
$h^* = \bar{h}$: grey; maximal fishing rate.
$h^* = 0.5$: hatched, intermediate fishing rate $\rightarrow$
*singular optimal control*.

# Fish dynamics and measure

$$U_T = DU_{XX} + R - \mu U - H(X)U, \quad -L/2 < X < L/2$$

$$\mathbf{BC}: U(-L/2, T) = U(L/2, T) = 0 \text{ for all } T \geq 0$$

- $DU_{XX}$: diffusion
- $R$: recruitment rate
- $\mu U$: natural death rate
- $H(X)U$: fishing rate

Find $H(X)$ in $[0, \bar{H}]$ that maximizes (at steady state):

$$J(H(X)) = \left(\frac{1}{L}\int_{-L/2}^{L/2} H(X)U(X)dX\right) + Q\left(\frac{1}{L}\int_{-L/2}^{L/2} U(X)dX\right)$$

# Scaling out parameters

leads to:

$$\begin{aligned} u' &= v \\ v' &= (1 + h(x))\, u - 1 \end{aligned}$$

$$u(-l/2) = u(l/2) = 0$$

Find $h(x)$ in $[0, \bar{h}]$ that maximizes:

$$j(h(x)) = \frac{1}{l} \int_{-l/2}^{l/2} (h(x) + q) u(x) dx$$

Only 3 parameters remain: $l, \bar{h}$ and $q$ (started with 6)

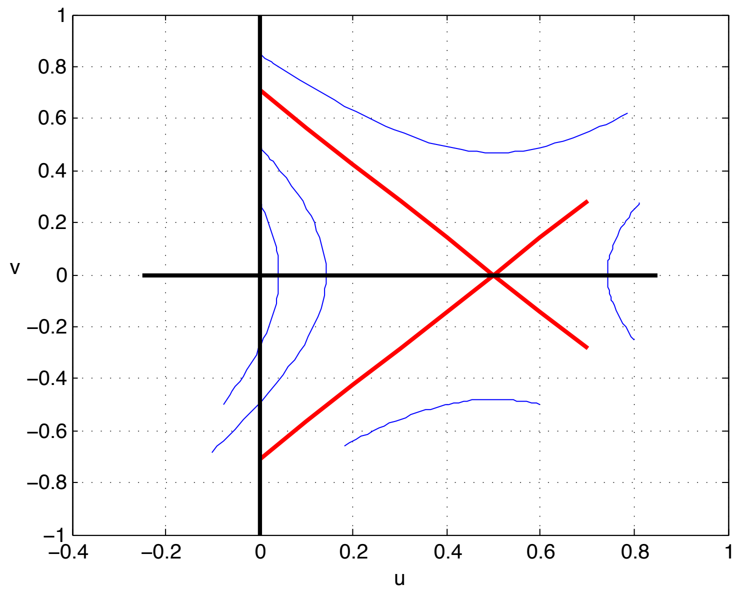
# Existence of optimal controls

Follows from Lee and Markus, *Foundations of optimal control*, 1967.

Main condition to check: existence of controls that solve the BVP (i.e. a controllability condition).

In fact, any constant control $h(x) = \widehat{h}$ with $\widehat{h} \in [0, \bar{h}]$ works.

$$
\begin{aligned}
u' &= v \\
v' &= \left(1+\widehat{h}\right)u - 1 \\
u(-l/2) &= u(l/2) = 0
\end{aligned}
$$



# Pontryagin maximum principle I

$$H(u, v, \lambda_1, \lambda_2, h) = \frac{1}{l}(h+q)u + \begin{pmatrix}\lambda_1 \\ \lambda_2\end{pmatrix} \cdot \begin{pmatrix} v \\ (1+h)u - 1\end{pmatrix}$$

$h^*(x)$ such that $\max_{h(x)} H(u^*(x), v^*(x), \lambda_1^*(x), \lambda_2^*(x), h(x))$

$$\Longrightarrow$$

$$h^*(x) = \begin{cases} \bar{h} \text{ if } u^*(x)\left(\frac{1}{l} + \lambda_2^*(x)\right) > 0 \\ 0 \text{ if } u^*(x)\left(\frac{1}{l} + \lambda_2^*(x)\right) < 0 \\ ? \text{ if } u^*(x)\left(\frac{1}{l} + \lambda_2^*(x)\right) = 0 \end{cases}$$

# Pontryagin maximum principle II

$$H(u, v, \lambda_1, \lambda_2, h) = \tfrac{1}{l}(h+q)u + \begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix} \cdot \begin{pmatrix} v \\ (1+h)u - 1\end{pmatrix}$$

$h^*(x)$ such that $\max_{h(x)} H(u^*(x), v^*(x), \lambda_1^*(x), \lambda_2^*(x), h(x))$

$$\Longrightarrow$$

$$h^*(x) = \begin{cases} \bar{h} \text{ if } u^*(x)\left(\frac{1}{l} + \lambda_2^*(x)\right) > 0 \\ 0 \text{ if } u^*(x)\left(\frac{1}{l} + \lambda_2^*(x)\right) < 0 \\ ? \text{ if } u^*(x)\left(\frac{1}{l} + \lambda_2^*(x)\right) = 0 \end{cases}$$

$u^*(x) > 0$ if $-l/2 < x < l/2$

# Pontryagin maximum principle III

$$H(u, v, \lambda_1, \lambda_2, h) = \frac{1}{l}(h+q)u + \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \cdot \begin{pmatrix} v \\ (1+h)u - 1 \end{pmatrix}$$

$h^*(x)$ such that $\max_{h(x)} H(u^*(x), v^*(x), \lambda_1^*(x), \lambda_2^*(x), h(x))$

$$\Longrightarrow$$

$$h^*(x) = \begin{cases} \bar{h} \text{ if } \lambda_2^*(x) > -\frac{1}{l} \\ 0 \text{ if } \lambda_2^*(x) < -\frac{1}{l} \\ ? \text{ if } \lambda_2^*(x) = -\frac{1}{l} \end{cases}$$

# Hamiltonian formulation

$$H(u,v,\lambda_1,\lambda_2,h) = \tfrac{1}{l}(h+q)u + \begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix} \cdot \begin{pmatrix} v \\ (1+h)u-1 \end{pmatrix}$$

Then $(u^*(x), v^*(x), \lambda_2^*(x), \lambda_2^*(x), h^*(x))$ must satsify:

$$\begin{aligned}
u' &= \partial H/\partial \lambda_1 = v \\
v' &= \partial H/\partial \lambda_2 = (1+h(x))u - 1 \\
\lambda_1' &= -\partial H/\partial u = -(h(x)+1)\lambda_2 - \frac{h(x)+q}{l} \\
\lambda_2' &= -\partial H/\partial v = -\lambda_1
\end{aligned}$$

$$\begin{aligned}
\mathbf{BC} : u(-l/2) &= u(l/2) = 0 \\
\mathbf{TC} : \lambda_2(-l/2) &= \lambda_2(l/2) = 0
\end{aligned}$$

# Adjoint System

$$
\begin{aligned}
\lambda_1' &= -(h(x)+1)\lambda_2 - \frac{h(x)+q}{l} \\
\lambda_2' &= -\lambda_1
\end{aligned}
$$

$$
\mathbf{TC}: \lambda_2(-l/2) = \lambda_2(l/2) = 0
$$

$$
h(x) = \begin{cases} \bar{h} \text{ if } \lambda_2(x) > -\frac{1}{l} \\ 0 \text{ if } \lambda_2(x) < -\frac{1}{l} \\ ? \text{ if } \lambda_2(x) = -\frac{1}{l} \end{cases}
$$

# Adjoint System

$$\begin{aligned}\lambda_1' &= -(h(x)+1)\lambda_2 - \frac{h(x)+q}{l} \\ \lambda_2' &= -\lambda_1\end{aligned}$$

$$\mathbf{TC}: \lambda_2(-l/2) = \lambda_2(l/2) = 0$$

$$h(x) = \begin{cases} \bar{h} \text{ if } \lambda_2(x) > -\frac{1}{l} \\ 0 \text{ if } \lambda_2(x) < -\frac{1}{l} \\ ? \text{ if } \lambda_2(x) = -\frac{1}{l} \end{cases}$$

Singular optimal control: does not occur

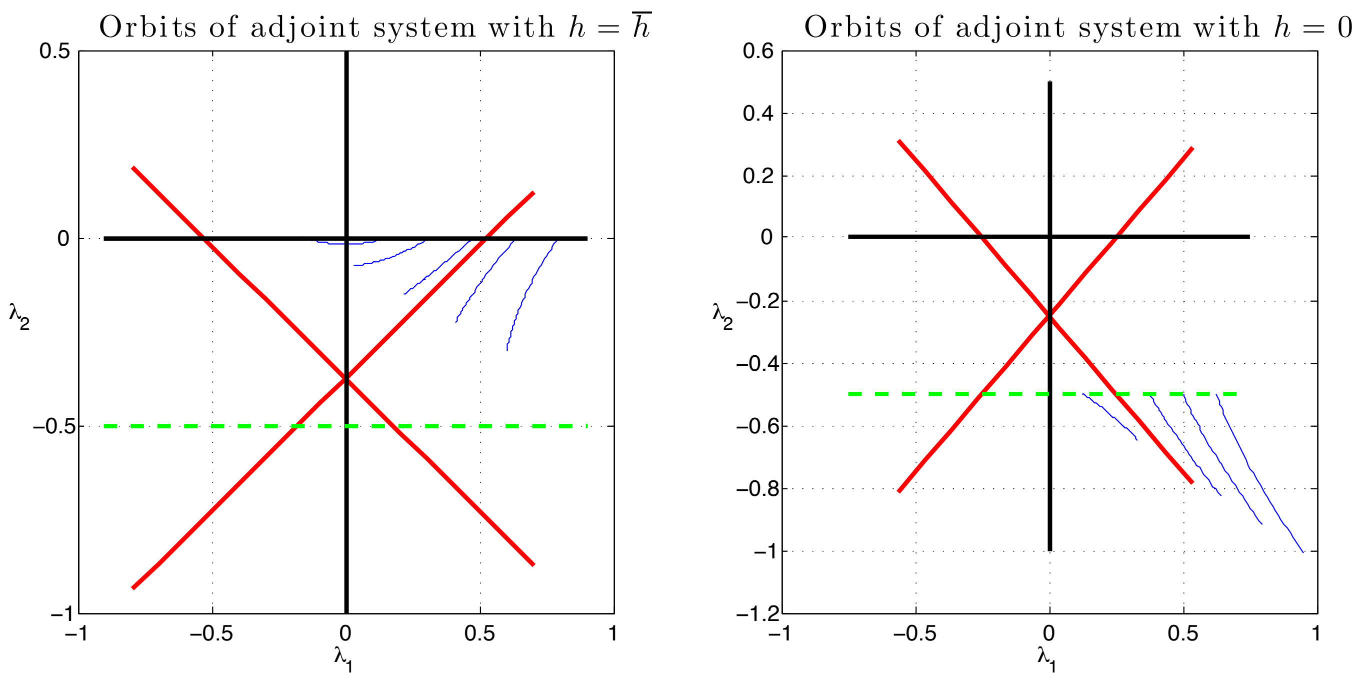
# Adjoint System

$$\begin{aligned}\lambda_1' &= -(h(x)+1)\lambda_2 - \frac{h(x)+q}{l} \\ \lambda_2' &= -\lambda_1\end{aligned}$$

$$\mathbf{TC}: \lambda_2(-l/2) = \lambda_2(l/2) = 0$$

$$h(x) = \begin{cases} \bar{h} \text{ if } \lambda_2(x) > -\frac{1}{l} \\ 0 \text{ if } \lambda_2(x) < -\frac{1}{l} \end{cases}$$

$$
\begin{aligned}
\lambda_1' &= -(h(x)+1)\lambda_2 - \frac{h(x)+q}{l} \\
\lambda_2' &= -\lambda_1
\end{aligned}
$$



# Adjoint System

$$\begin{aligned} \lambda_1' &= -(h(x)+1)\lambda_2 - \frac{h(x)+q}{l} \\ \lambda_2' &= -\lambda_1 \end{aligned}$$

$$\mathbf{TC}: \lambda_2(-l/2) = \lambda_2(l/2) = 0$$

$$h(x) = \begin{cases} \bar{h} \text{ if } \lambda_2(x) > -\frac{1}{l} \\ 0 \text{ if } \lambda_2(x) < -\frac{1}{l} \end{cases}$$

Symmetry property: reflection w.r.t $\lambda_2$-axis

$\Longrightarrow \ldots$

# Adjoint System

$$\begin{aligned}\lambda_1' &= -(h(x)+1)\lambda_2 - \frac{h(x)+q}{l}\\ \lambda_2' &= -\lambda_1\end{aligned}$$

$$\mathbf{TC}: \lambda_2(-l/2) = \lambda_1(0) = 0$$

$$h(x) = \begin{cases}\bar{h} \text{ if } \lambda_2(x) > -\frac{1}{l}\\ 0 \text{ if } \lambda_2(x) < -\frac{1}{l}\end{cases}$$

Modified 2nd TC

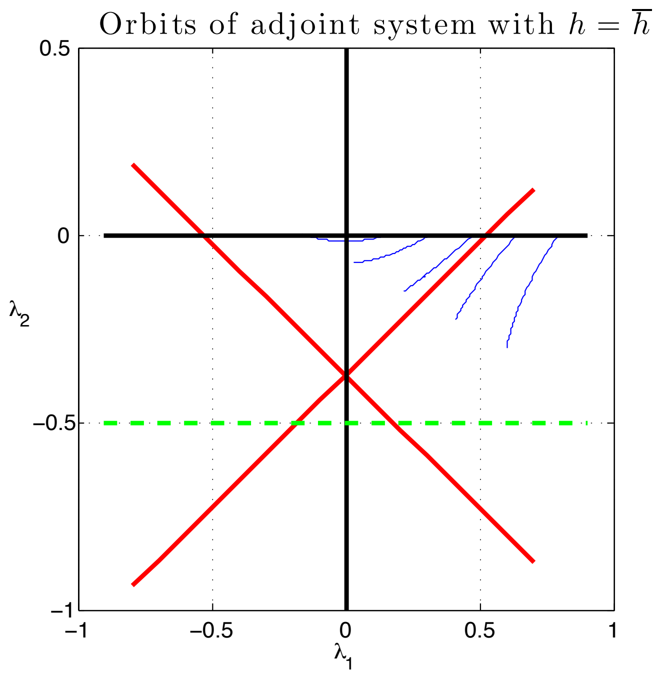
## Two cases: $q \leq 1$ and $q > 1$

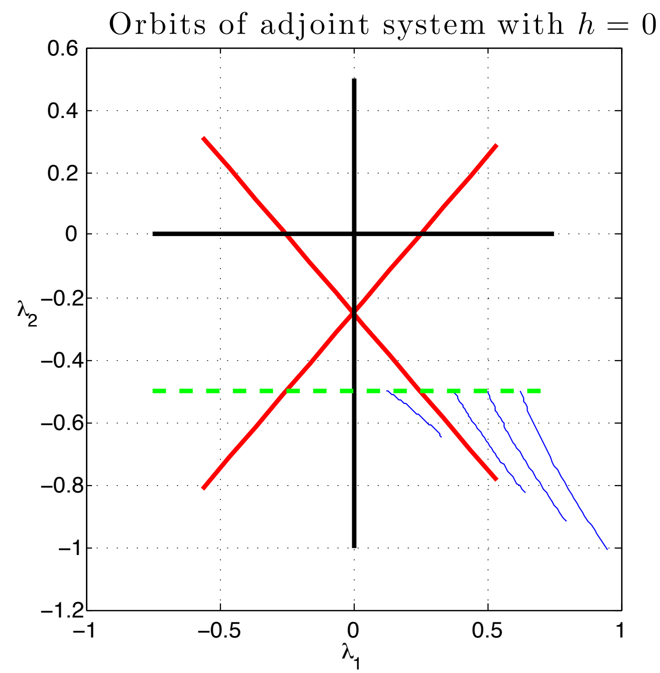
Lead to qualitatively different phase portraits for adjoint system:

1. $q \leq 1 \Longrightarrow$ no switch.

2. $q > 1 \Longrightarrow$ switch may or may not occur.

# Case $q \leq 1$




No switch since then $\lambda_1(0) = 0$ would be impossible.

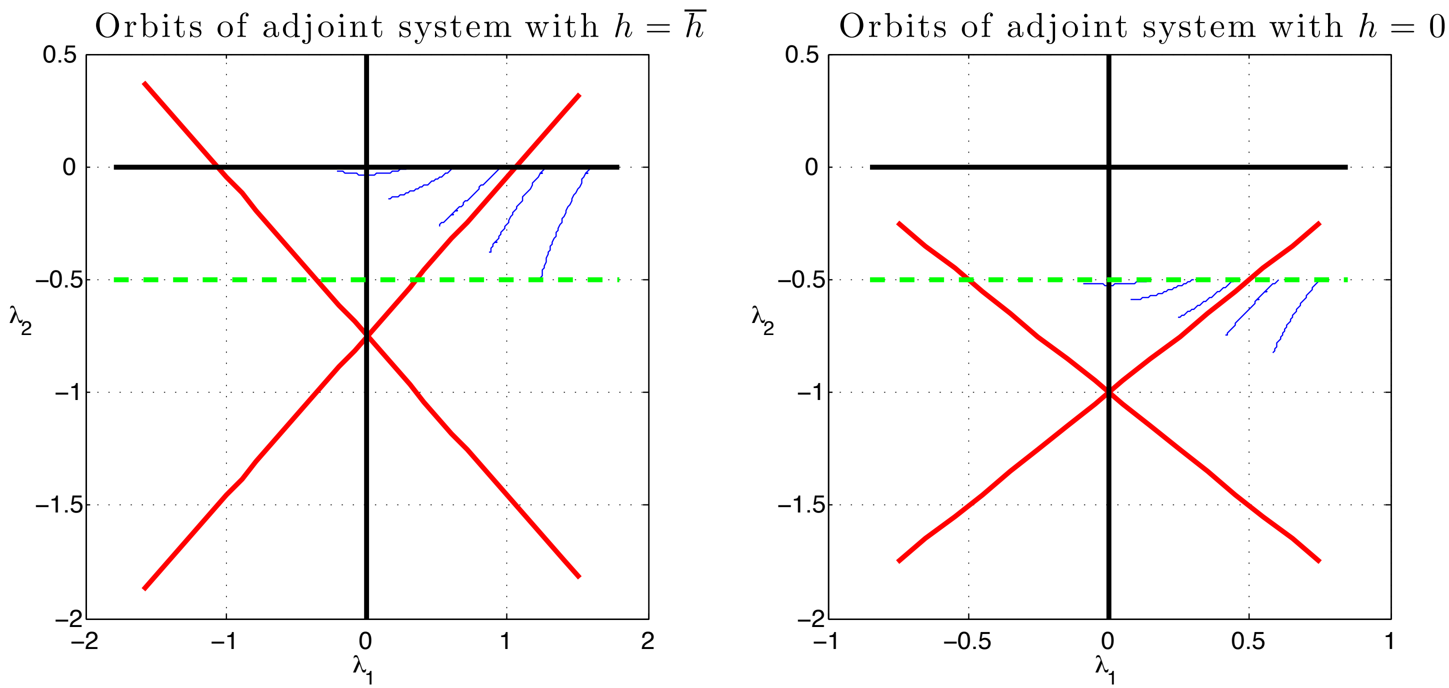
# Case $q \leq 1$

$$h^*(x) = \bar{h} \text{ for all } x \in [-l/2, l/2]$$

Fishing everywhere at maximal rate $\Longrightarrow$ No MPA.

**Note**: $q \leq 1$ means conservationists' impact is small.

# Case $q > 1$




Switch may occur, but when?

# Case $q > 1$

Solving:

$$\begin{aligned}\lambda_1' &= -(h(x)+1)\lambda_2 - \frac{h(x)+q}{l}\\ \lambda_2' &= -\lambda_1\end{aligned}$$

$$\lambda_1(-l/2) = \lambda_0 \ (> 0 \text{ parameter}), \lambda_2(-l/2) = 0$$

$$h(x) = \begin{cases}\bar{h} \text{ if } \lambda_2(x) > -\frac{1}{l}\\ 0 \text{ if } \lambda_2(x) < -\frac{1}{l}\end{cases}$$

## Case $q > 1$

Determine "when" (i.e. $x$-value) solution first hits $\lambda_2$-axis:

$$T_0(\lambda_0) = \inf\{x > -l/2 | \lambda_1(x) = 0\}$$

Note: $T_0 = +\infty$ if solution never hits $\lambda_2$-axis.

# Case $q > 1$

Properties of $T_0$:

- $\lim_{\lambda_0 \to 0} T_0(\lambda_0) = 0$

- $\lim_{\lambda_0 \to \lambda_0^{**}} T_0(\lambda_0) = +\infty$. [Define $\lambda_0^{**}$]

- $T_0$ is increasing.

$\Longrightarrow$ There is a unique $\bar{\lambda}_0 \in (0, \lambda_0^{**})$: $T_0(\bar{\lambda}_0) = l/2$.

# Relevance of $\bar{\lambda}_0$ (cont.)

Value of $\bar{\lambda}_0$ determines whether a switch occurs or not:

Switch iff $\bar{\lambda}_0 > \lambda_0^*$.

[Define $\lambda_0^*$]

Equivalently, applying the increasing function $T_0$:

Switch iff

$$l/2 > T_0(\lambda_0^*) \equiv \frac{1}{\sqrt{\bar{h}+1}} \operatorname{arctanh}\left(\frac{\sqrt{(\bar{h}+1)(\bar{h}+2q-1)}}{\bar{h}+q}\right)$$

# Case $q > 1$

Let $l_{\text{min}} = \dfrac{2}{\sqrt{\bar{h}+1}} \operatorname{arctanh}\left(\dfrac{\sqrt{(\bar{h}+1)(\bar{h}+2q-1)}}{\bar{h}+q}\right)$

- If $l \le l_{\text{min}}$, then $h^*(x) = \bar{h}$ for all $x \Longrightarrow$ No MPA
- If $l > l_{\text{min}}$, then

$$h^*(x) = \begin{cases} 0 \text{ if } x \in [-b, b] \Longrightarrow \text{Single MPA} \\ \bar{h} \text{ if } x \in [-l/2, -b) \text{ or } (b, l/2] \end{cases}$$

[Define $b$*]

*Implicit formula available, but not shown

# Summary

| $\mathbf{l \backslash q}$ | $q \leq 1$ | $q > 1$ |
|---|---|---|
| $l \leq l_{\text{min}}$ | no MPA | no MPA |
| $l > l_{\text{min}}$ | no MPA | 1 MPA in middle, size $2b$ |

If conservationist pressure high (large $q$) and coastline long (large $l$), then establish 1 MPA.

Otherwise: don't.

# Thank You!