Complex dynamics caused by facilitation in polluted landscapes

MIAMI 2012-12-15

Dedicated to C. Cosner for his 60th birthday

J. López-Gómez (UCM), M. Molina-Meyer (UC3M), A. Tellini (UCM) and F. Zanolin (UniUd)

A parabolic problem

Analyzing the dynamics of the positive solutions of

$$\begin{cases} \partial_t u = \Delta u + \lambda u + a(x)\, u^p & \text{in } \Omega, \quad t > 0, \\ u = M & \text{on } \partial\Omega, \quad t > 0, \\ u(x,0) = u_0(x) & \text{if } x \in \Omega. \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is smooth and bounded, $M > 0$, $u_0 > 0$, $\lambda < 0$ and $p > 1$.

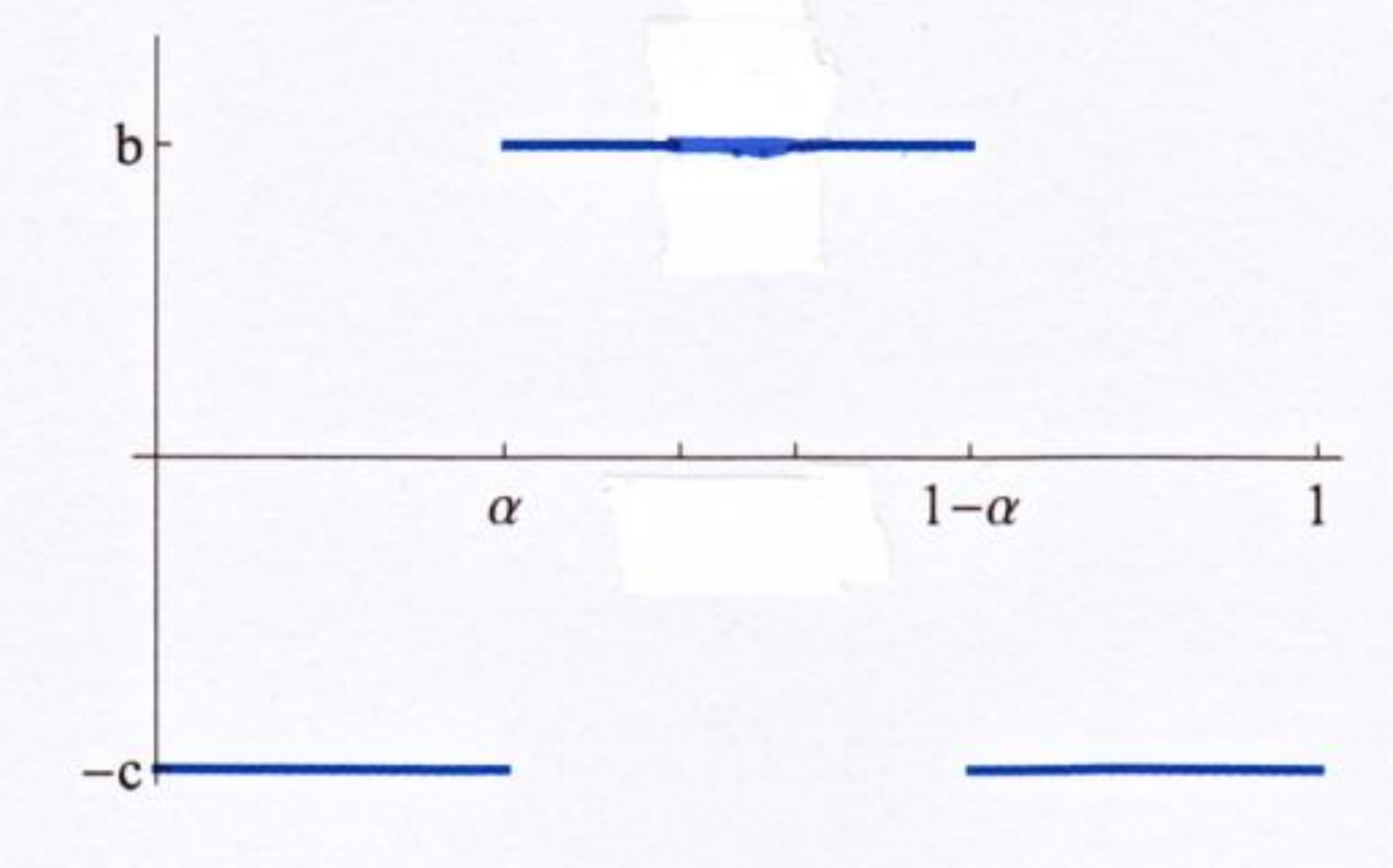
Polluted lanscape is measured by $\lambda < 0$ and $M > 0$

Nodal behavior of $a(x)$

- $a(x) > 0$ if $x \in \Omega_+$ with $\Omega_+ \Subset \Omega$,
- $a(x) < 0$ if $x \in \Omega_-$ $= \Omega - \overline{\Omega_+}$

In a neighborhood of $\partial\Omega$ the problem is sublinear, while it is superlinear in Ω_+.

A paradigmatic example



Structure of the talk

1. **Analysis of the one-dimensional model.**
2. **Global bifurcation diagrams using b as the main parameter for $\lambda \to -\infty$.**
3. **Numerical computations of the global bifurcation diagrams. Strong squashing effects for $\lambda \to -\infty$.**
4. **Uniqueness of the stable solution …**

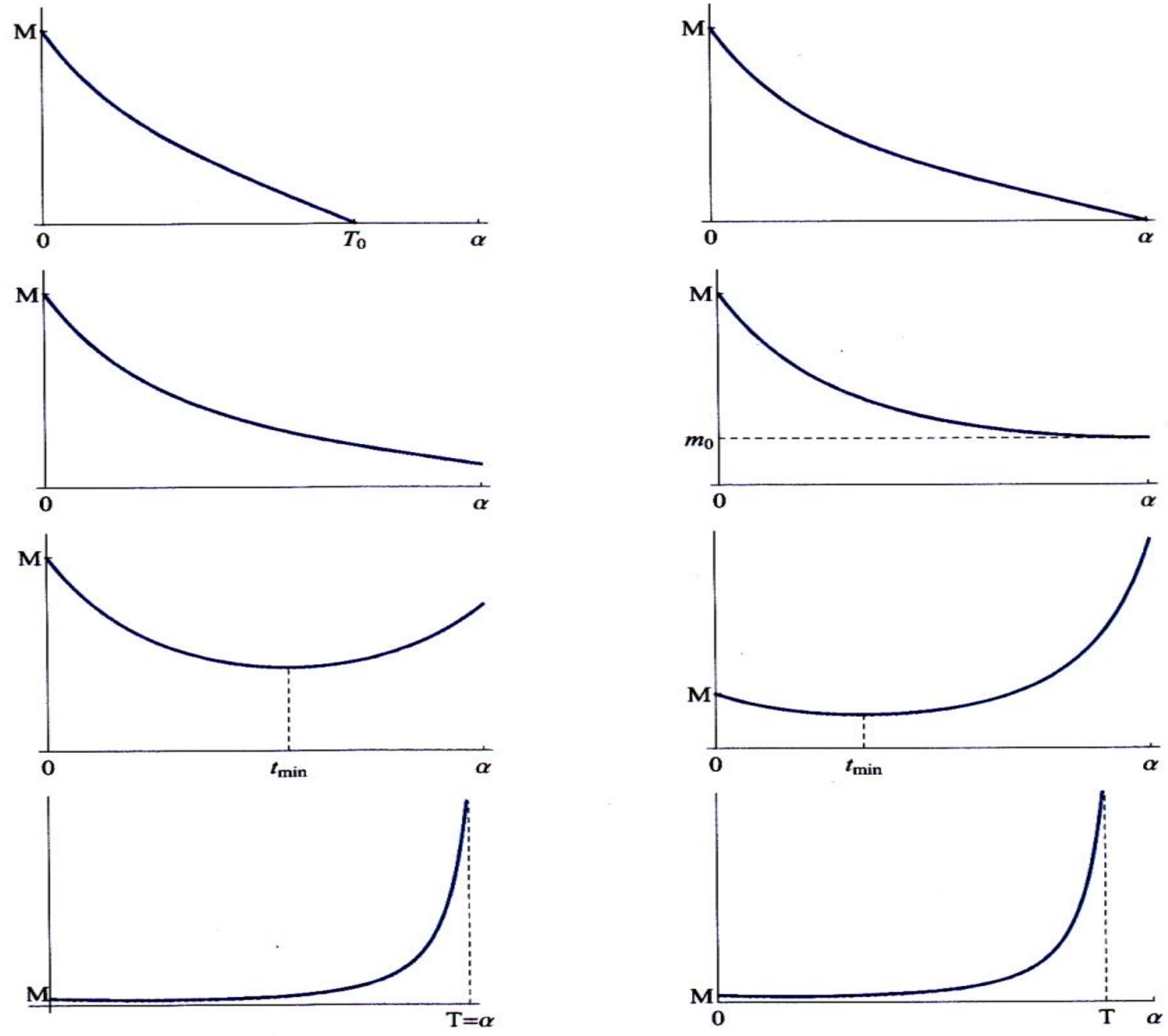
1. The one-dimensional model

Analysis of the sublinear problems in the intervals $[\mathbf{0}, \boldsymbol{\alpha}]$ and $[\mathbf{1} - \boldsymbol{\alpha}, \mathbf{1}]$, i.e.,

$$-u'' = \lambda u - cu^p \qquad \textbf{in } [\mathbf{0}, \boldsymbol{\alpha}],$$
$$u(0) = M, \qquad u'(0) = v \in \mathbb{R},$$

and

$$-u'' = \lambda u - cu^p \qquad \textbf{in } [\mathbf{1} - \boldsymbol{\alpha}, \mathbf{1}],$$
$$u(1) = M, \qquad u'(1) = v \in \mathbb{R}.$$


M
0
T_0
α
M
0
α
M
0
α
M
m_0
0
α
M
0
$t_{\min}$
α
M
0
$t_{\min}$
α
M
$T=\alpha$
M
0
T
α

The set reached at time α

Let $\boldsymbol{v}_*$ denote the unique value of v for which $\boldsymbol{u}(\boldsymbol{\alpha}) = \mathbf{0}$.

Let $\boldsymbol{v}^*$ denote the unique value of v for which $\boldsymbol{u}(\boldsymbol{\alpha}) = \infty$.

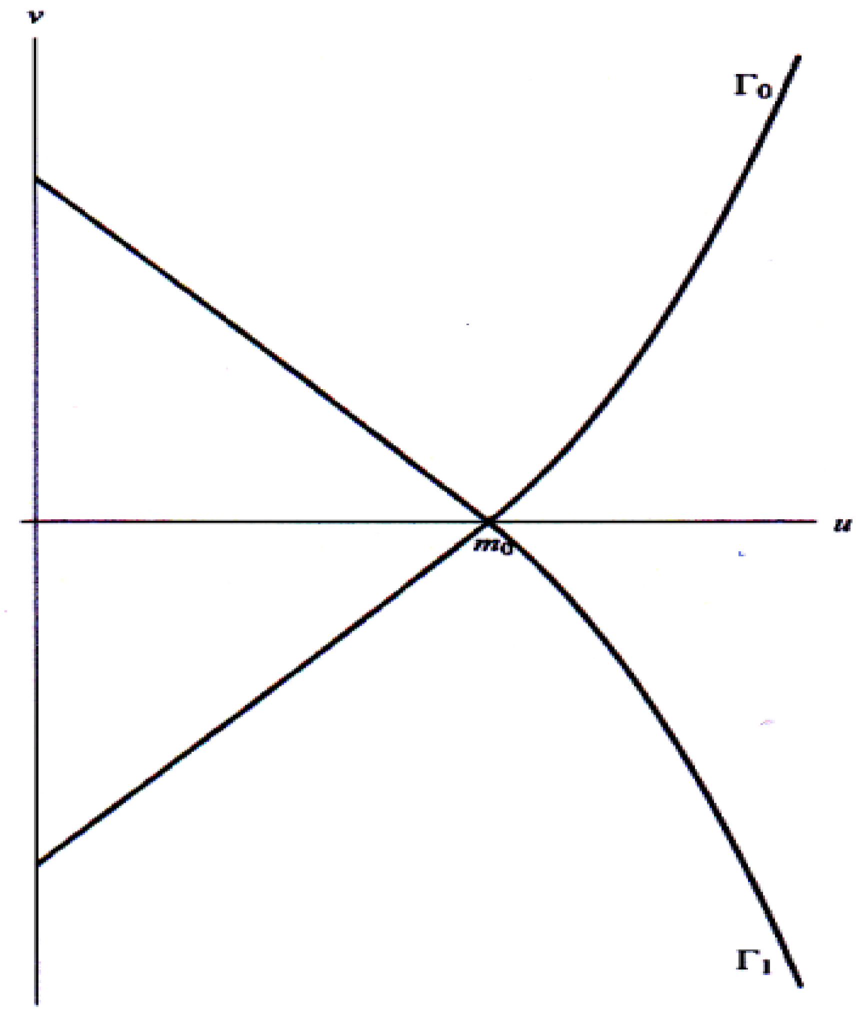
Then, we consider the curves of $\mathbb{R}^+ \times \mathbb{R}$

$$\Gamma_0 = \{ \left(u(\alpha), u'(\alpha)\right) : v \in [v_*, v^*) \}$$

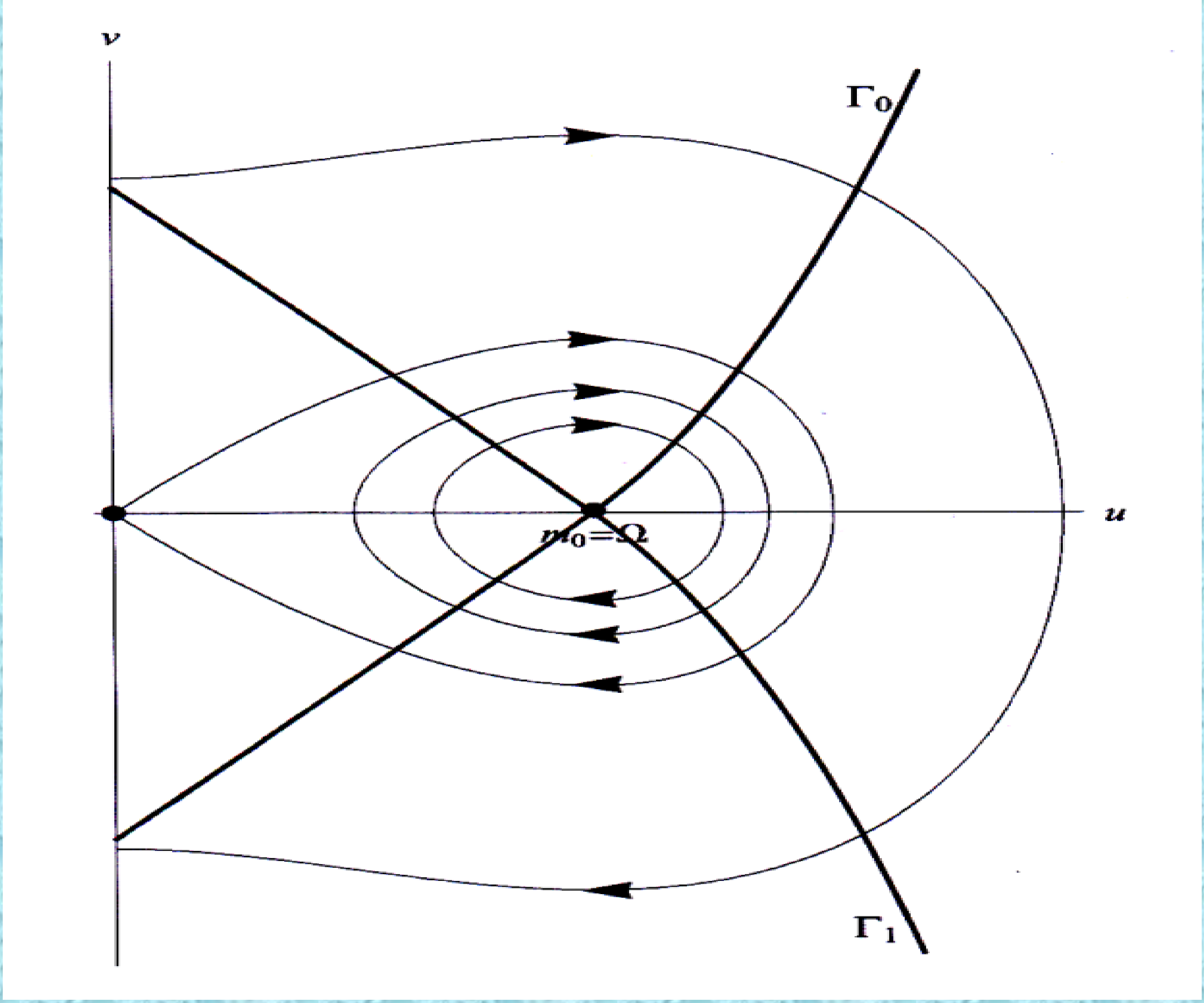
and

$$\Gamma_1 = \{ \left(u(1-\alpha), u'(1-\alpha)\right) : v \in -[v_*, v^*) \}$$

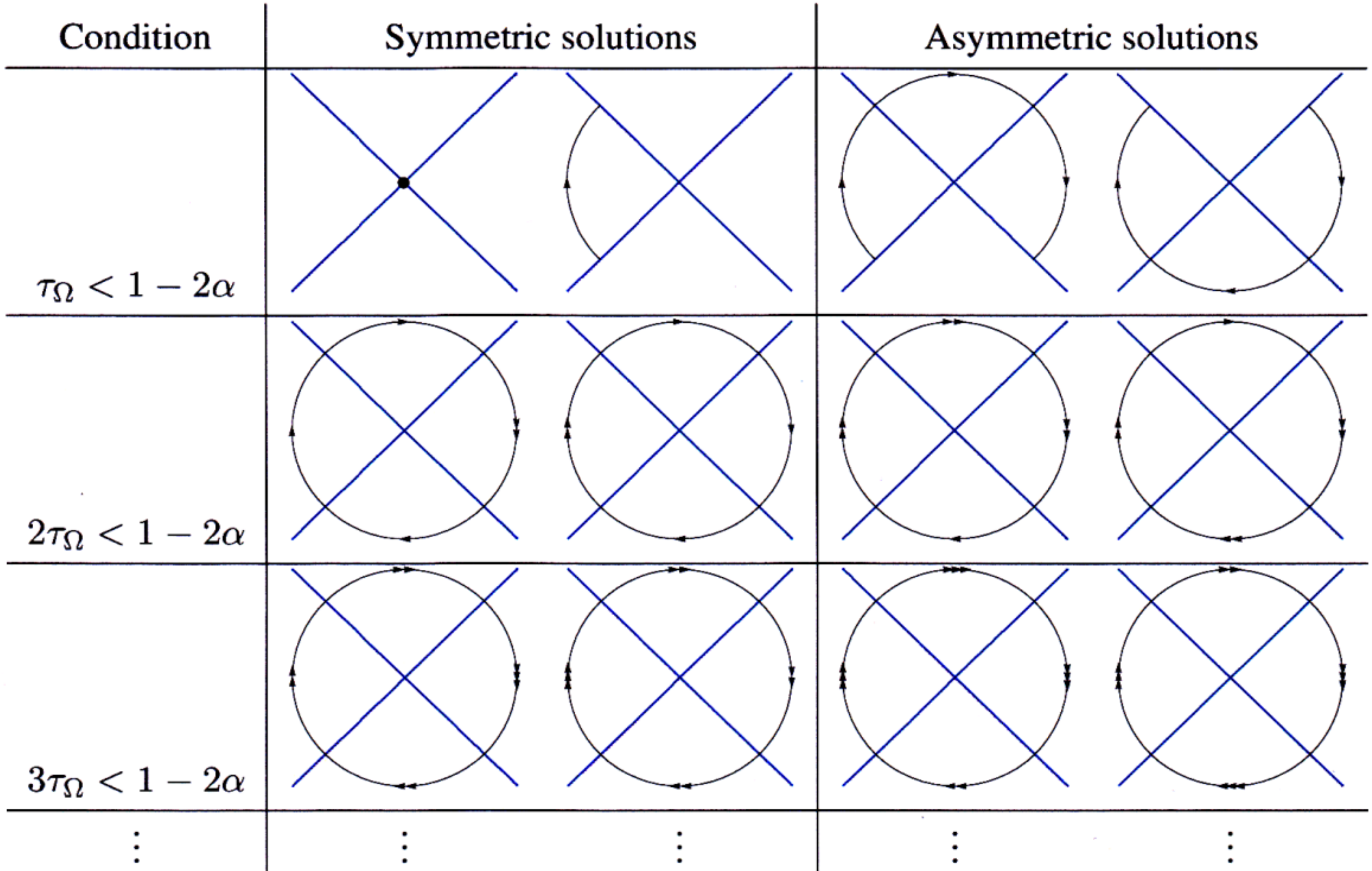
$\mathbb{\Gamma}_1 = \{(u, -v) : (u, v) \in \mathbb{\Gamma}_0\}$



The special case $b = b^*$=-$\lambda/{m_0}^{p-1}$



Multiplicity result $\left(\tau_\Omega = \frac{2\pi}{\sqrt{\lambda(1-p)}}\right)$



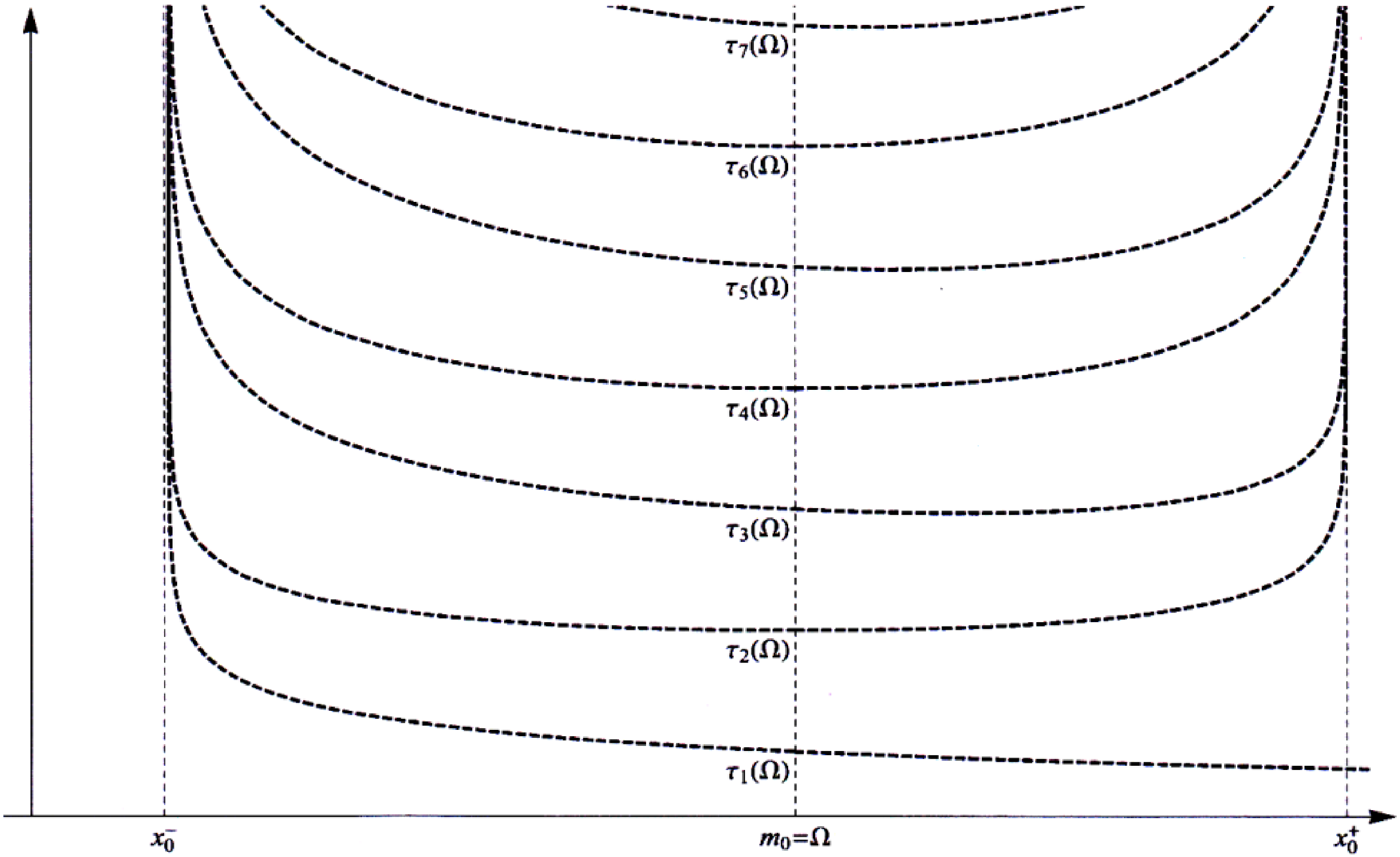
The Poincaré maps for $b = b^*$

- $\boldsymbol{\tau_1(x)}$ **is the minimal time needed to reach** $\mathbb{\Gamma}_{\mathbf{1}}$ **starting at** $(\boldsymbol{x}, \boldsymbol{y}(\boldsymbol{x})) \in \mathbb{\Gamma}_{\mathbf{0}}$.

- $\boldsymbol{\tau_{2n+1} = \tau_1 + n\tau}$, $\quad \boldsymbol{\tau_{2n} = \tau_1 + (n-1)\tau}$.

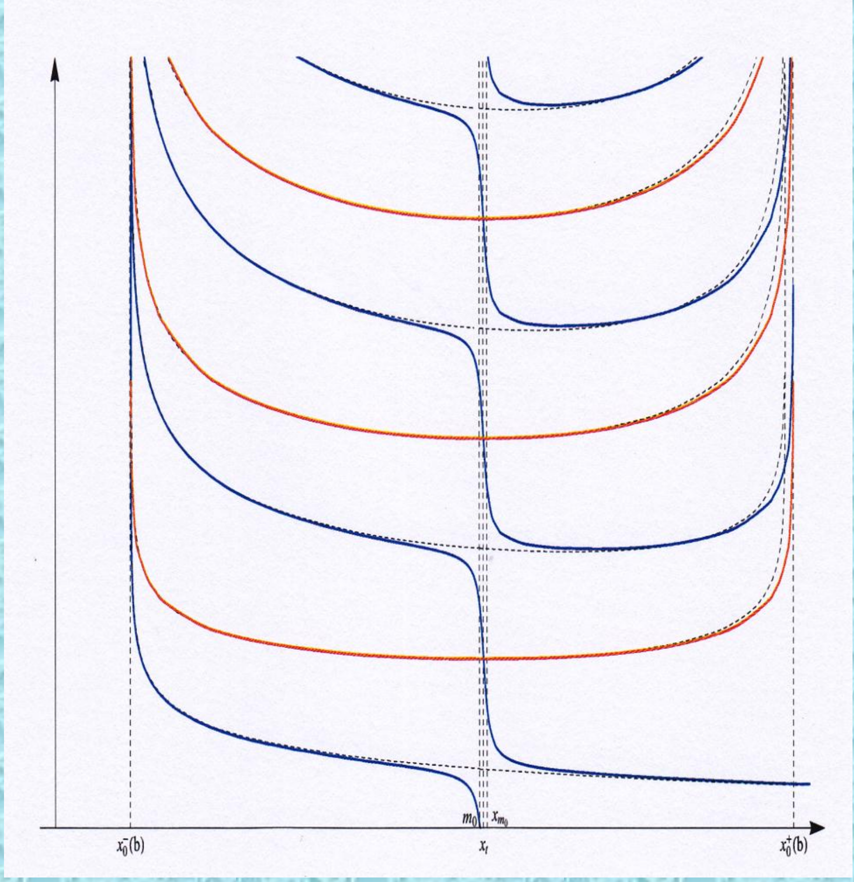
- $\boldsymbol{\tau(x)}$ **stands for the period of the solution starting at** $(\boldsymbol{x}, \boldsymbol{y}(\boldsymbol{x})) \in \mathbb{\Gamma}_{\mathbf{0}}$.

Graphs of the Poincaré maps

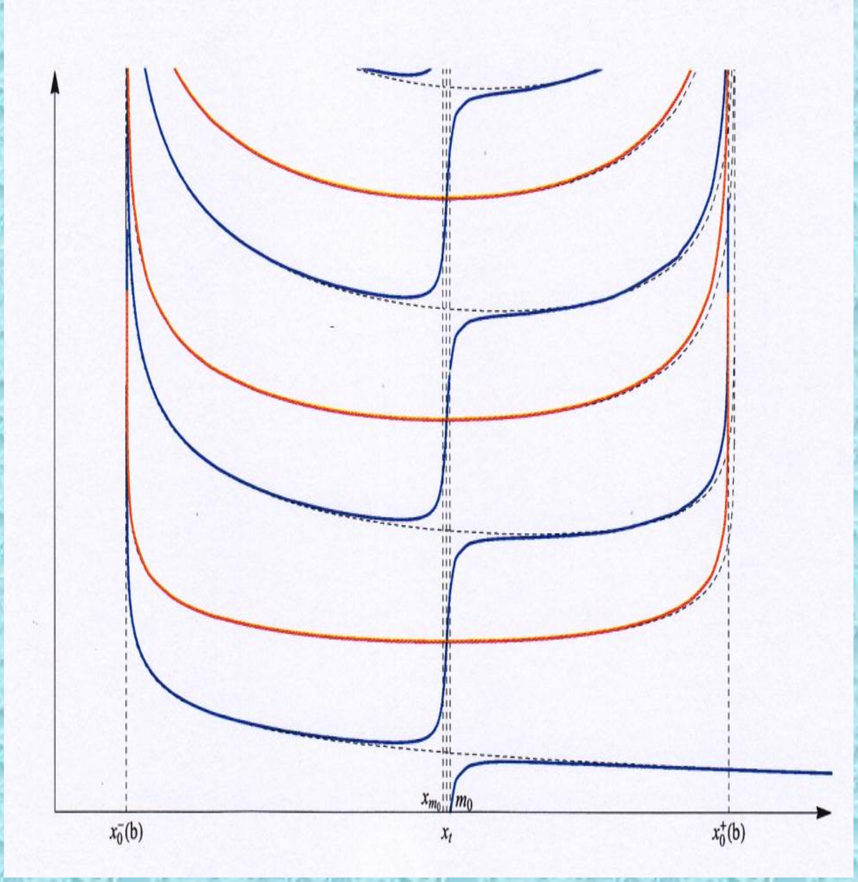


Perturbed Poincaré maps for $b \neq b^*$

$\boldsymbol{b < b^*}$

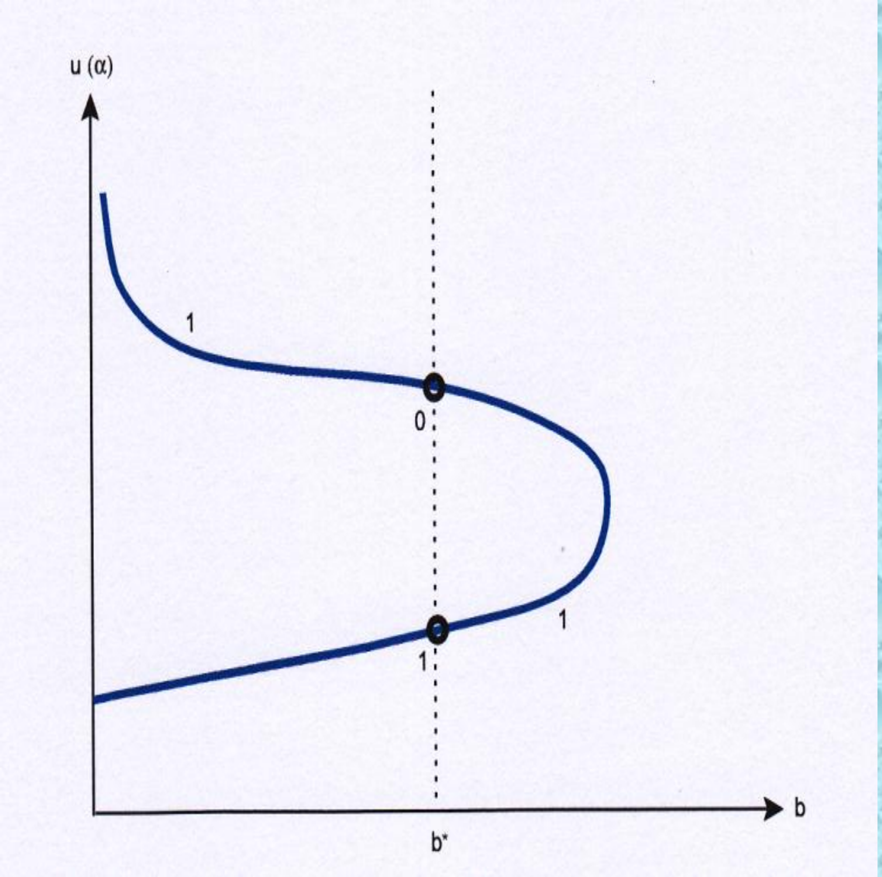


$\boldsymbol{b^* < b}$

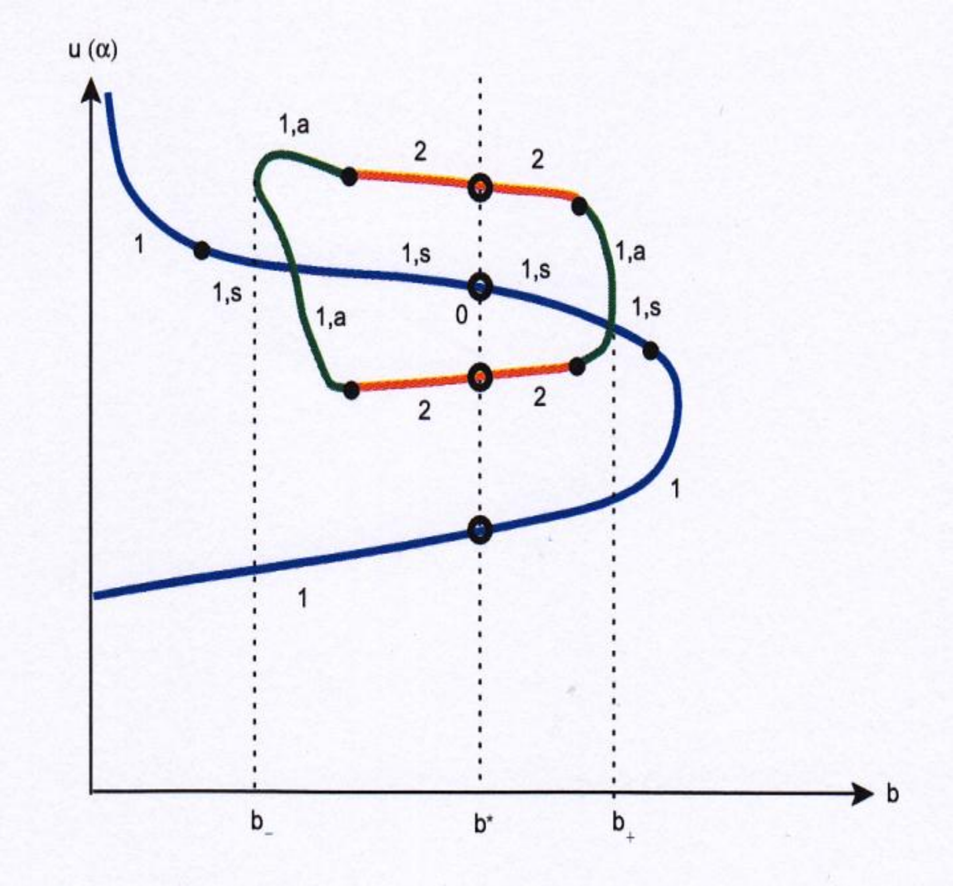


2. Global bifurcation diagrams in b

$\boldsymbol{\tau_1}(\Omega)$ **< 1 - 2**$\boldsymbol{\alpha} < \boldsymbol{\tau_2}(\Omega)$

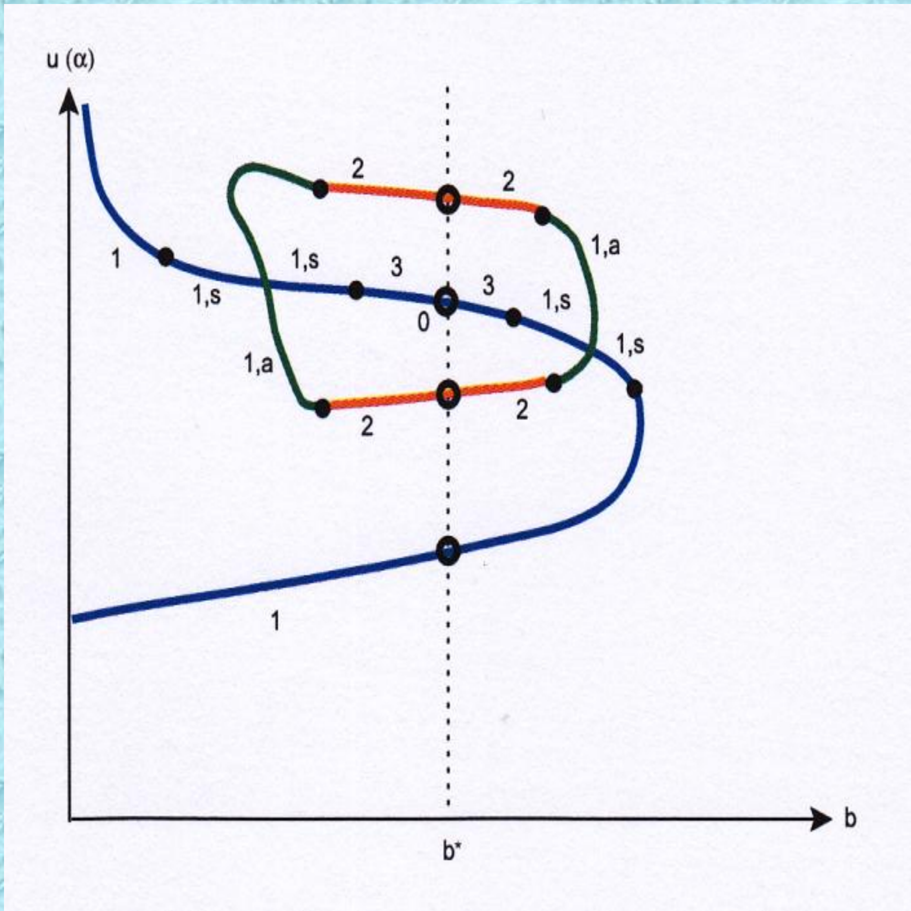


$\boldsymbol{\tau_2}(\Omega) < \mathbf{1} - \mathbf{2}\boldsymbol{\alpha} < \boldsymbol{\tau}(\Omega)$

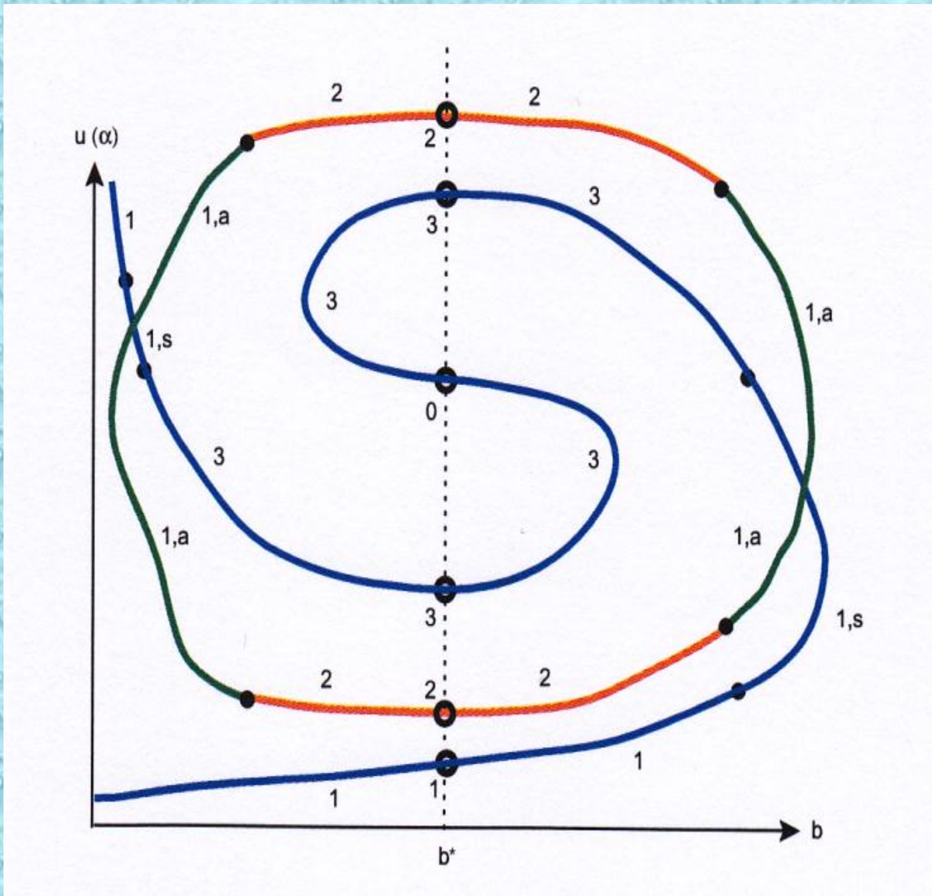


Emergence of loops and torsions

$\boldsymbol{\tau}(\Omega) < \mathbf{1} - \mathbf{2}\alpha < \boldsymbol{\tau}_{\mathbf{3}}(\Omega)$

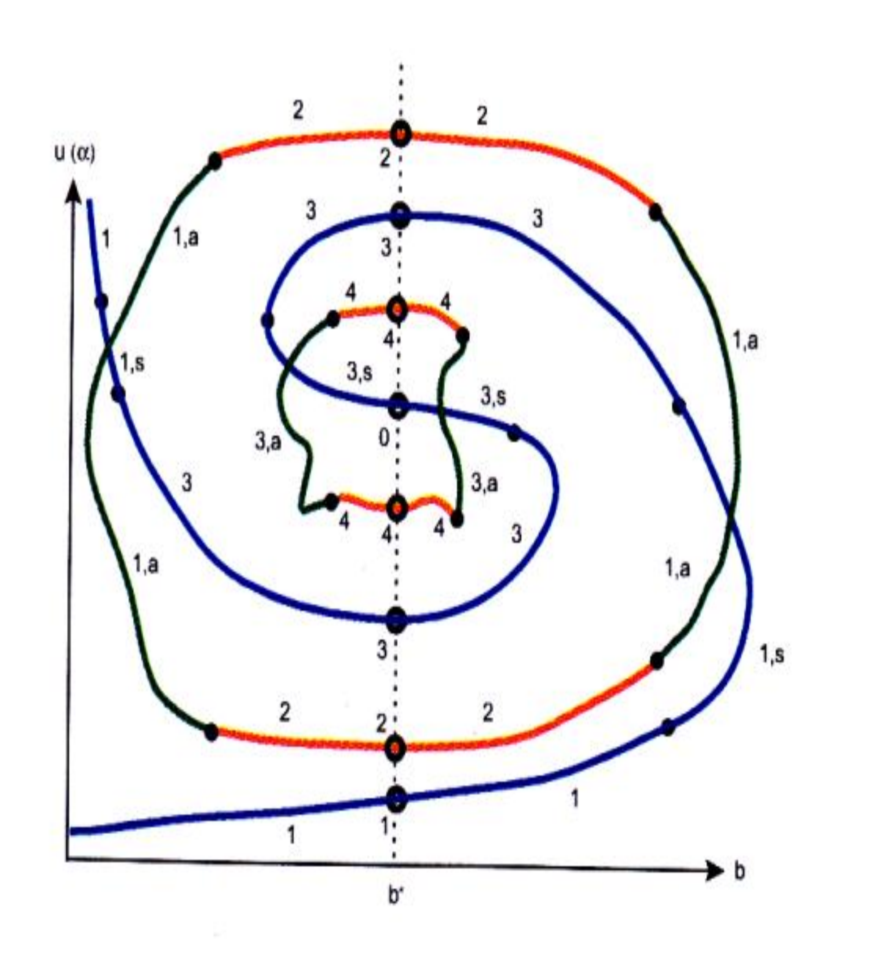


$\boldsymbol{\tau}_{\mathbf{3}}(\Omega) < \mathbf{1} - \mathbf{2}\boldsymbol{\alpha} < \boldsymbol{\tau}_{\mathbf{4}}(\Omega)$

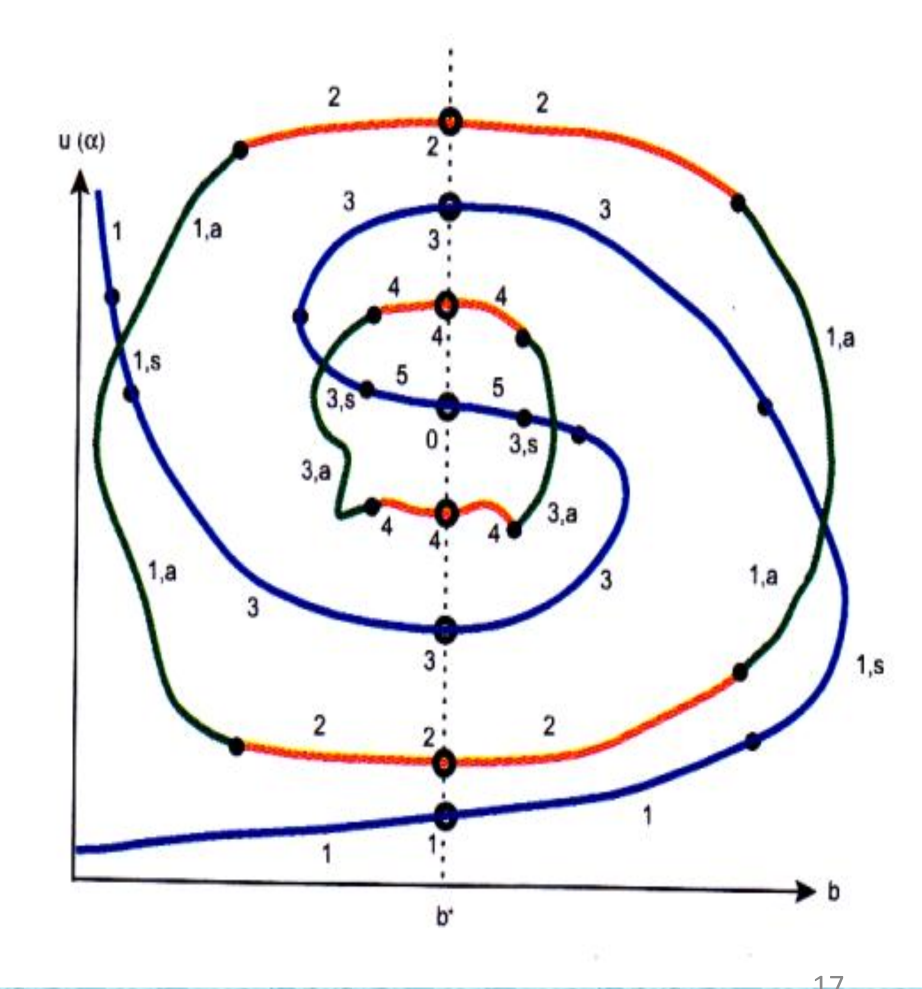


Secondary loops

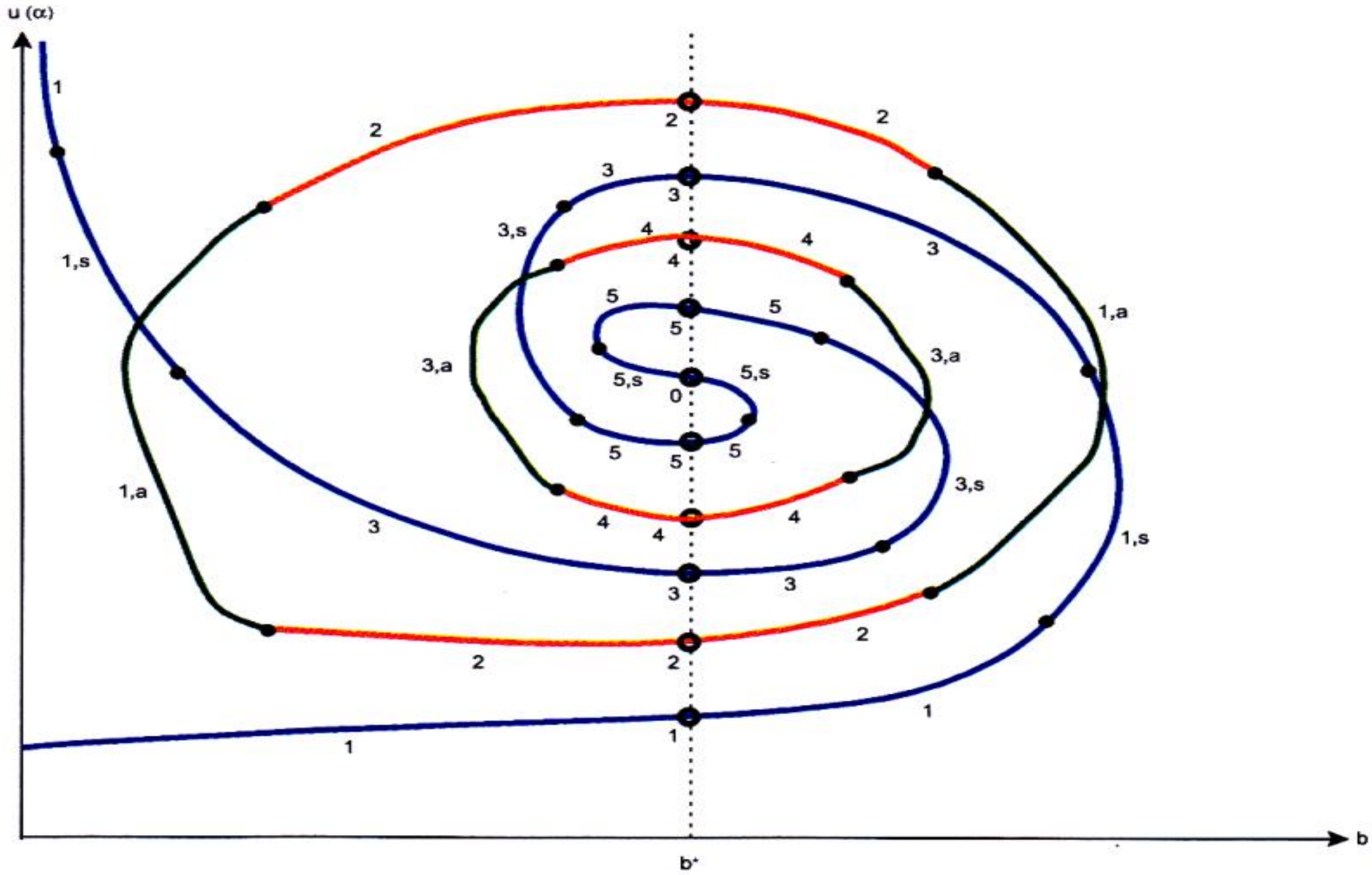
$\tau_4(\Omega) < 1 - 2\alpha < 2\tau(\Omega)$



$2\tau(\Omega) < 1 - 2\alpha < \tau_5(\Omega)$



Secondary torsion



3. Computing the bifurcation diagrams

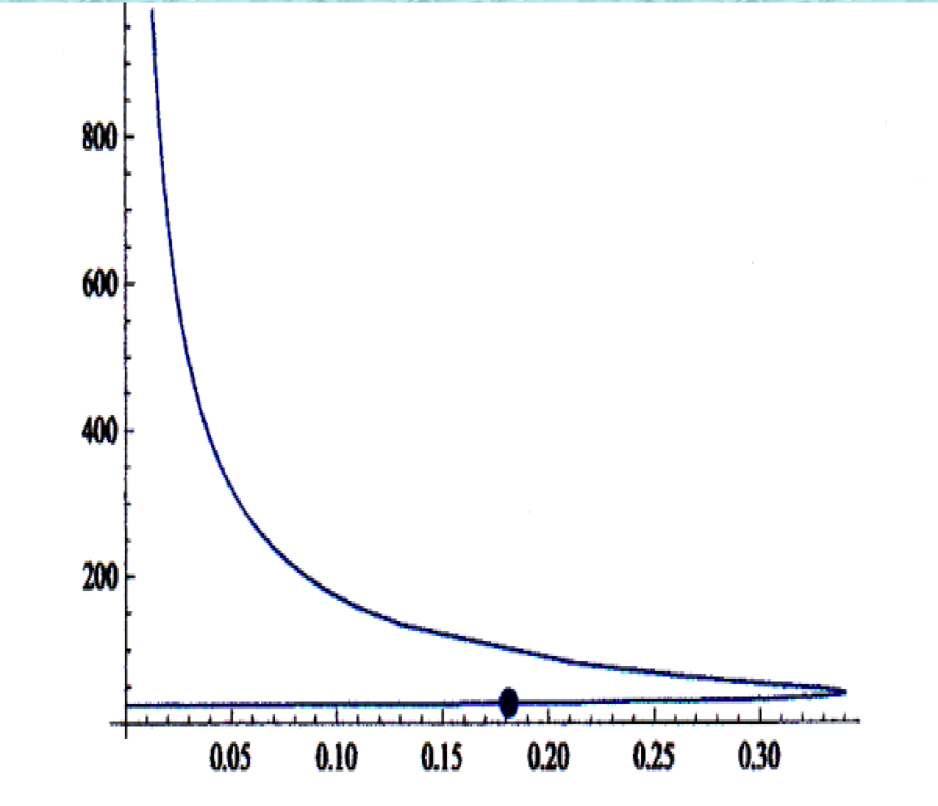
- **Discretization through spectral methods coupled with collocation**

- **Global continuation through local and global path-following solvers. Stability**

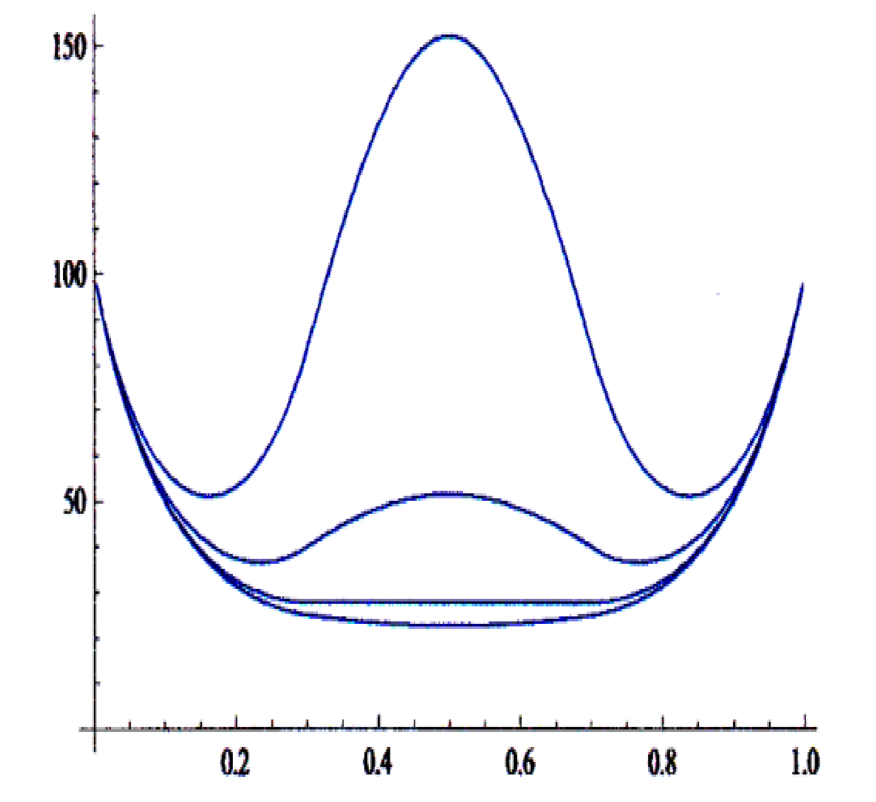
- **Strong squashing effects as** $\lambda \to -\infty$

$p = 2; M = 100; c = 1; \alpha = 0.3$

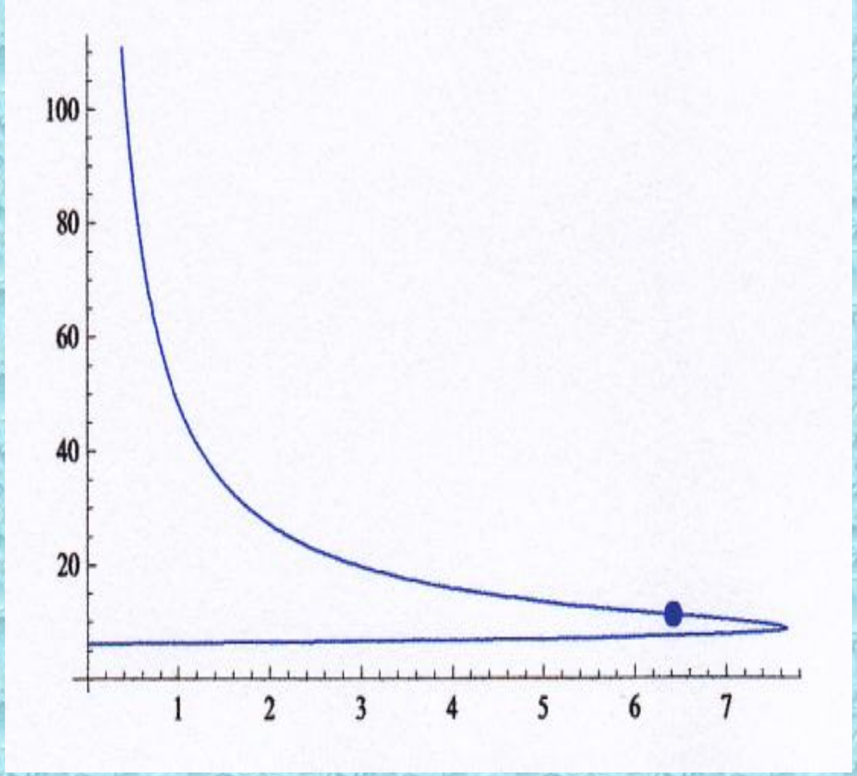
Diagram for $\lambda = -5$

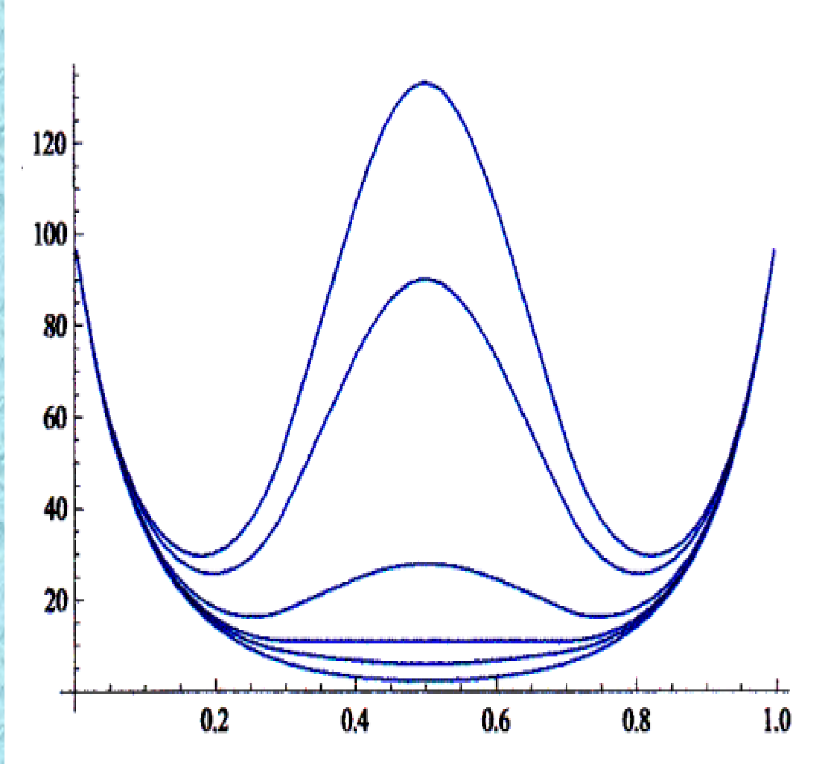


Some solutions along it

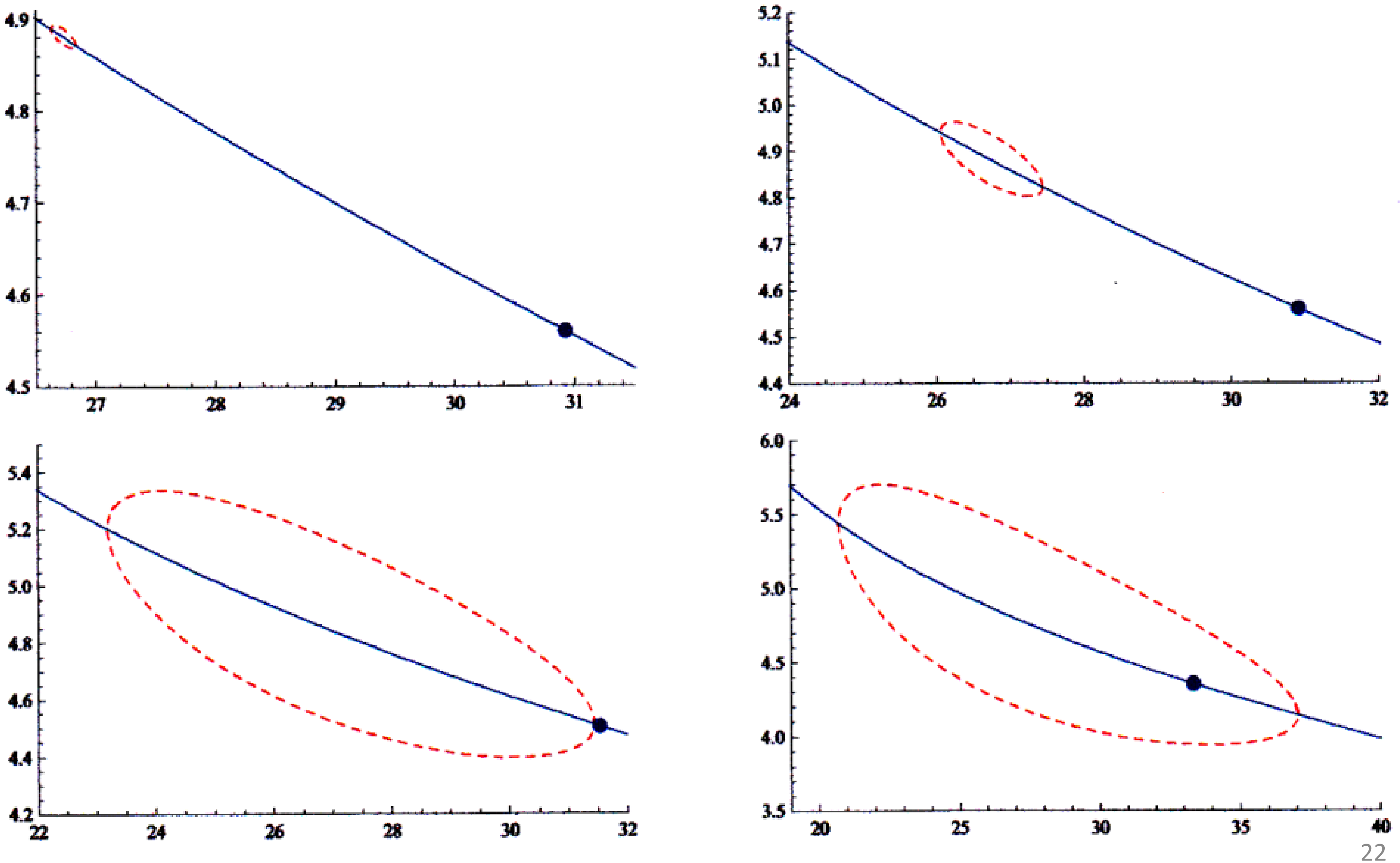


Bifurcation diagram for $\lambda = -70$

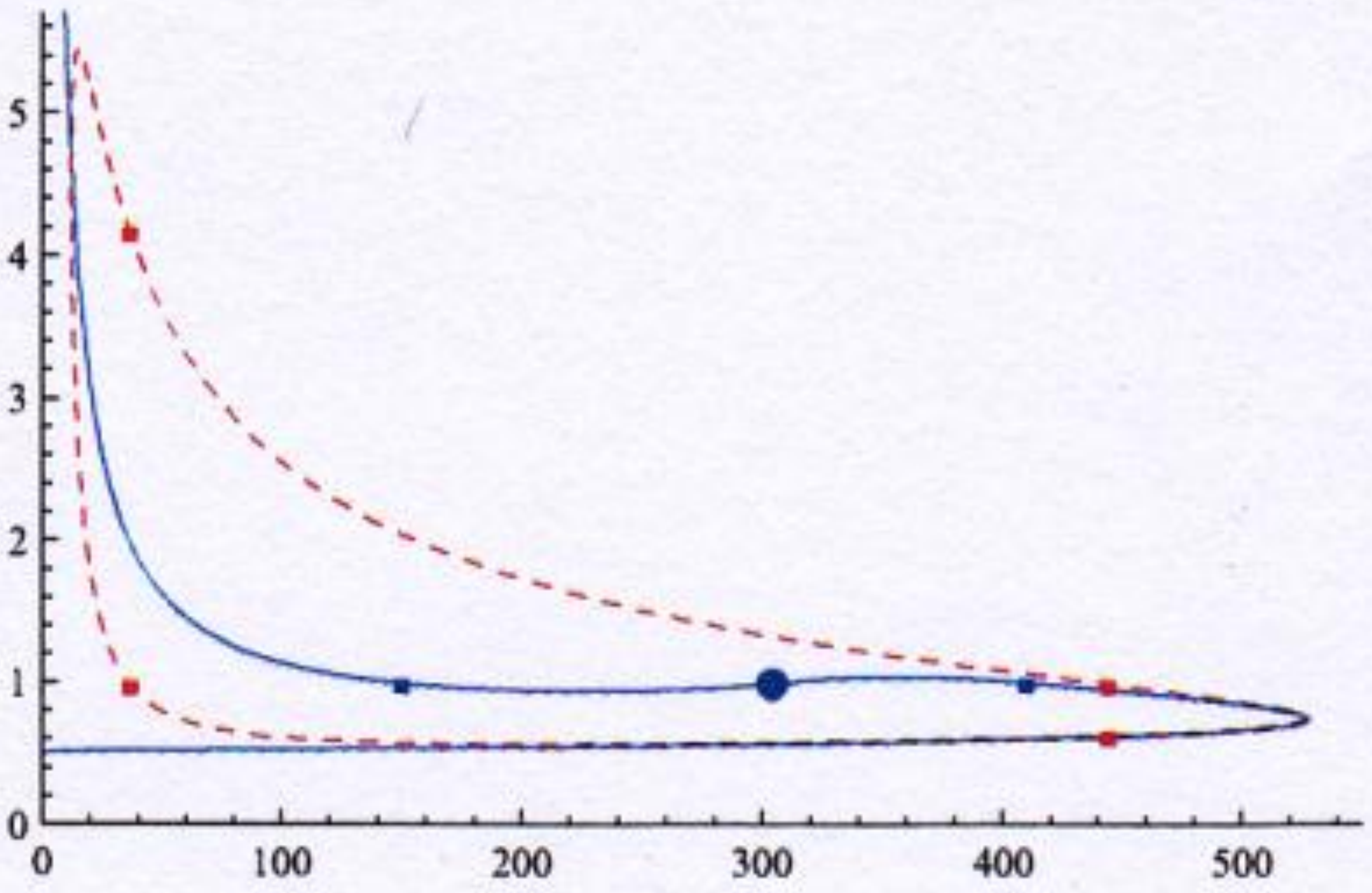




Symmetry breaking: The first loop



Bifurcation diagram for $\lambda = -300$



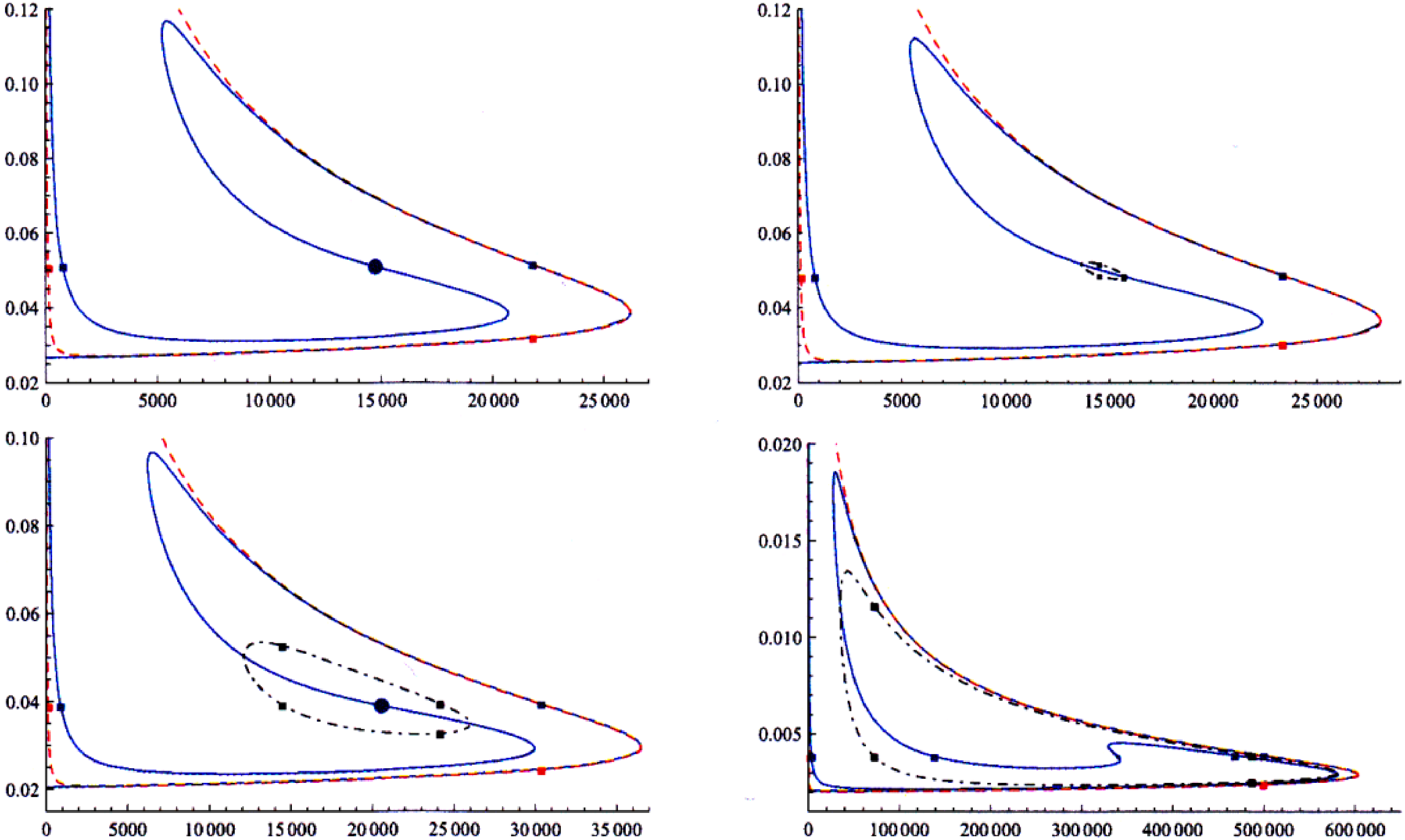


FIGURE 7. Global bifurcation diagram for $\lambda = -750$ (upper left), $\lambda = -760.3$ (upper right), $\lambda = -800$ (bottom left) and $\lambda = -1300$ (bottom right)

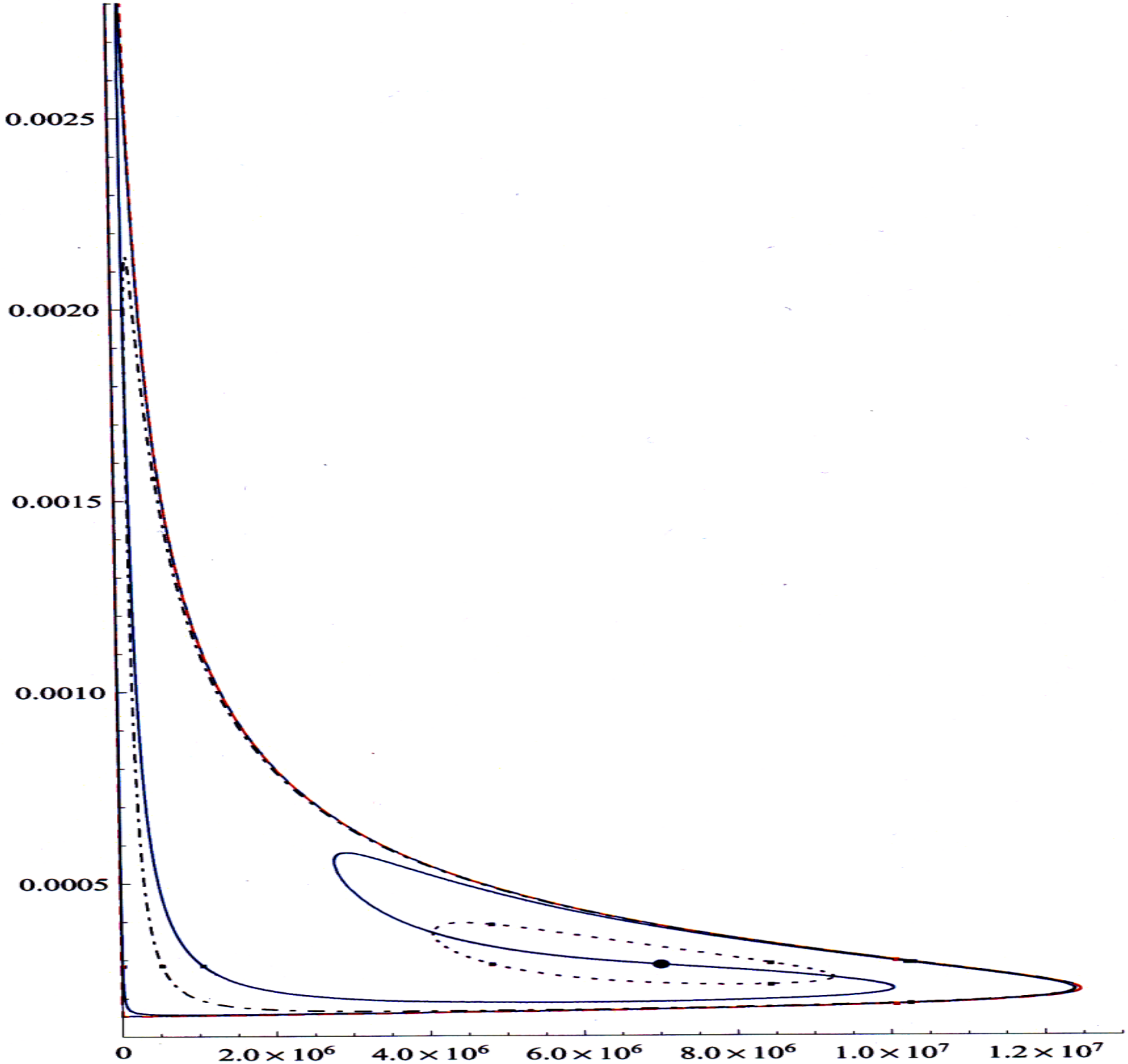
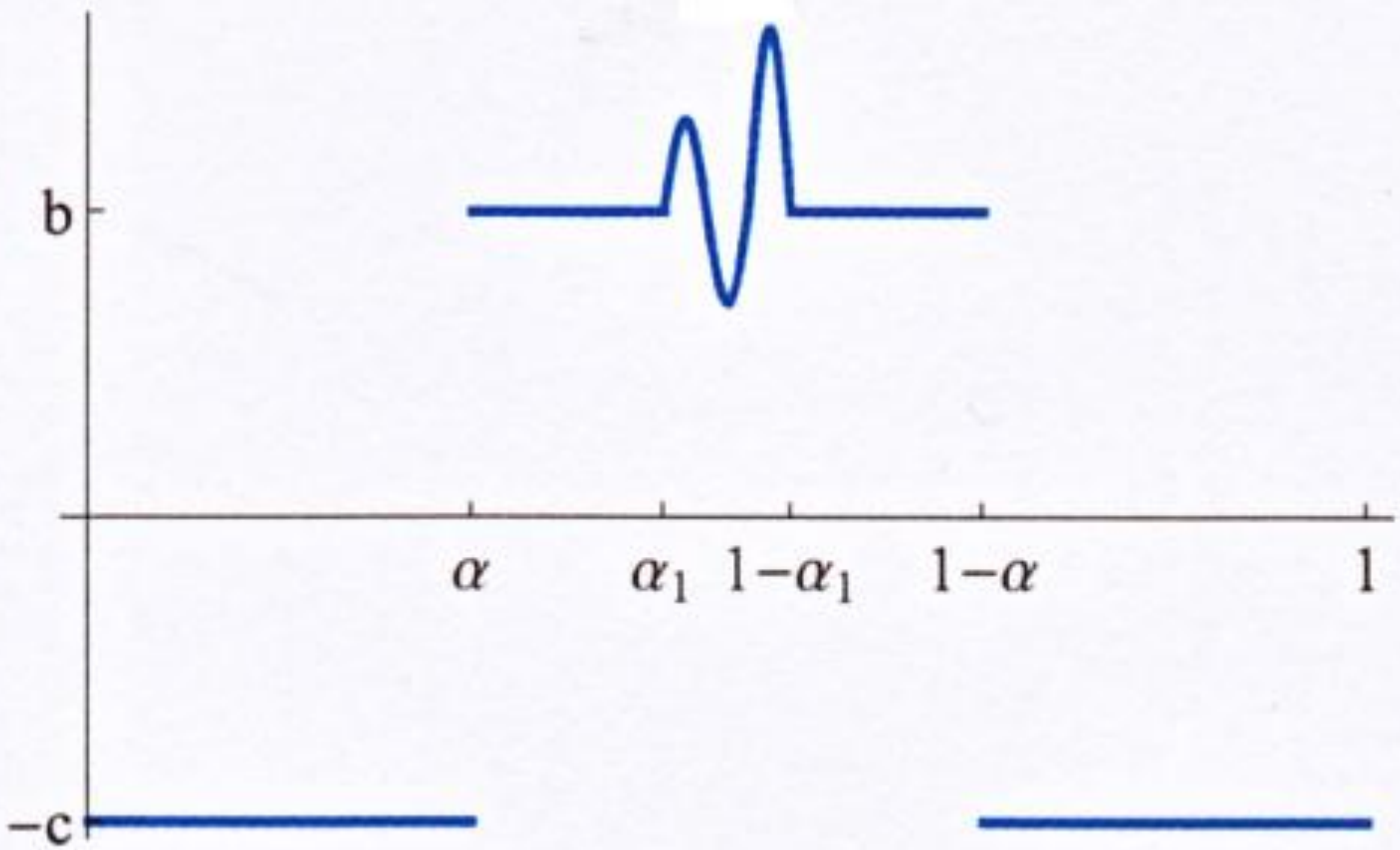
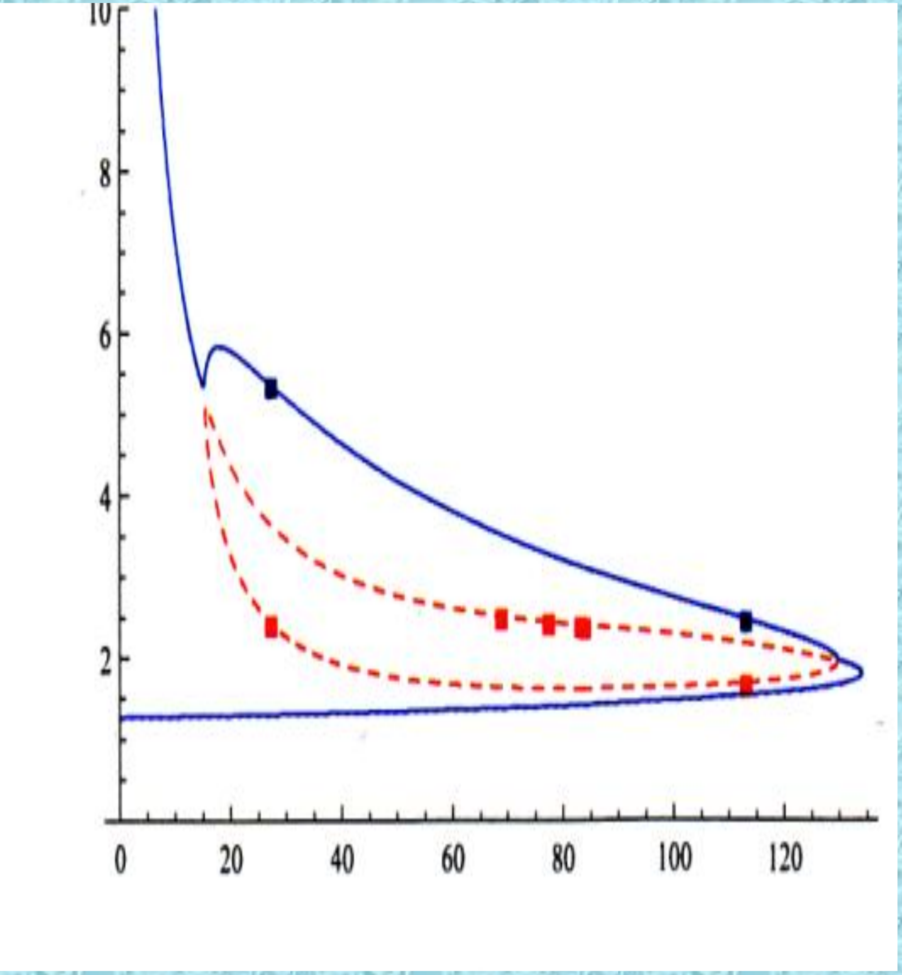


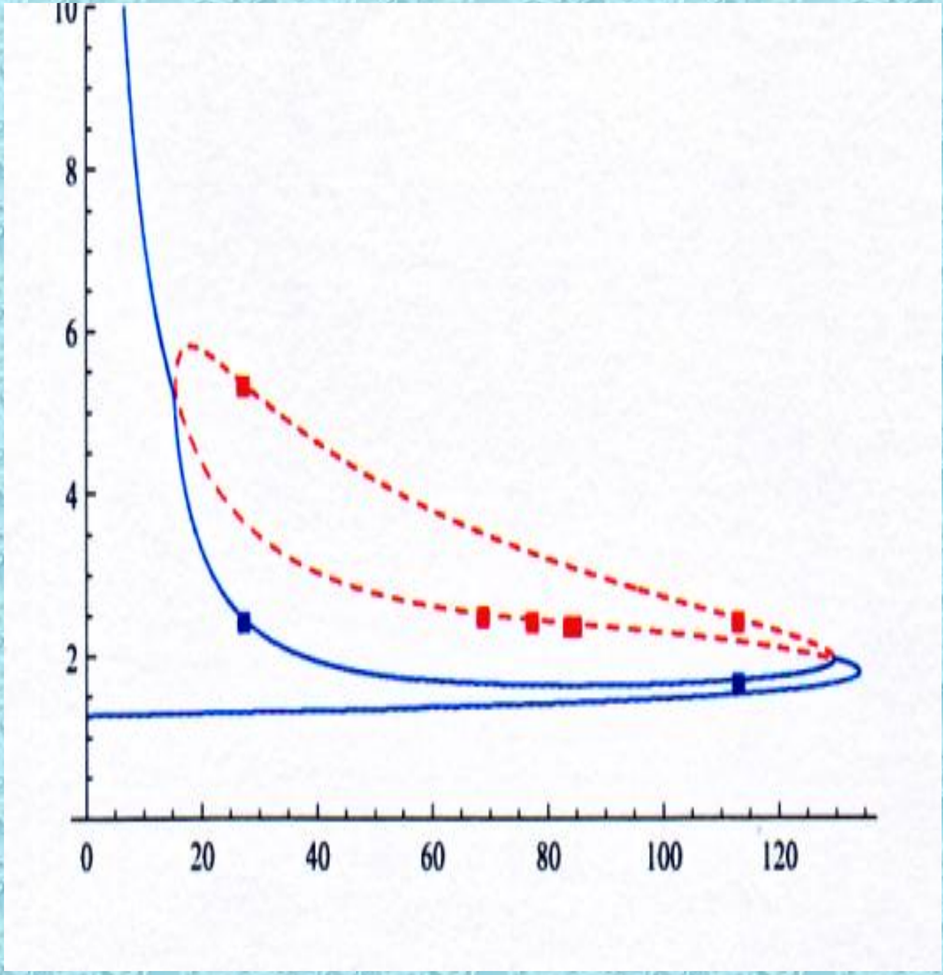
FIGURE 9. Global bifurcation diagram for $\lambda = -2000$

Asymmetric perturbation of $a(x)$

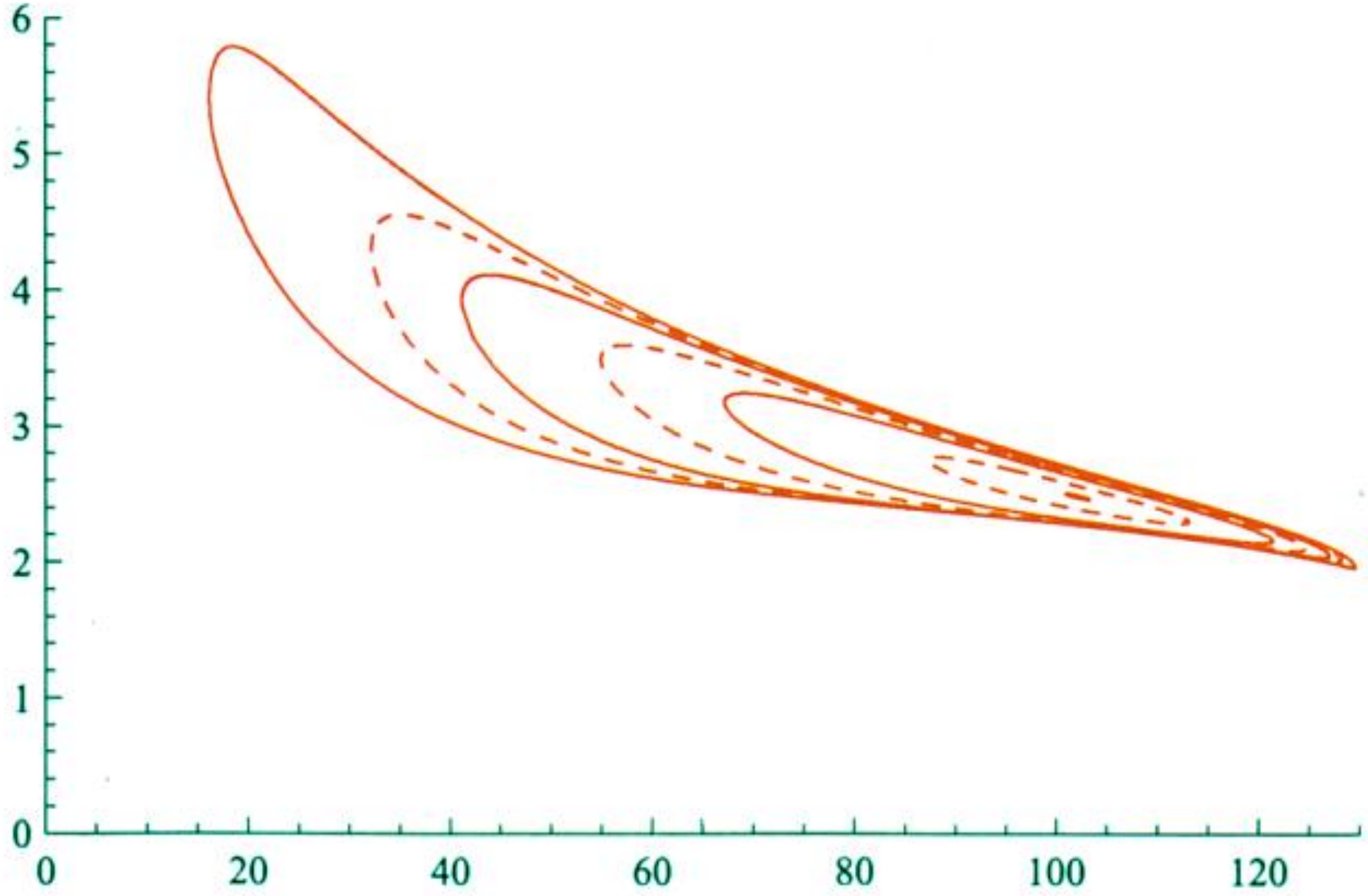


Two solution components





A family of isolas shrinking to a point



4. The general problem

(a) The problem does not admit a positive steady-state for sufficiently large $b > 0$.

(b) The minimal positive steady-state is the unique stable steady-state.

(c) In the presence of a priori bounds, the model possesses at least two solutions, except at the maximal value of b.

Stable and unstable solutions

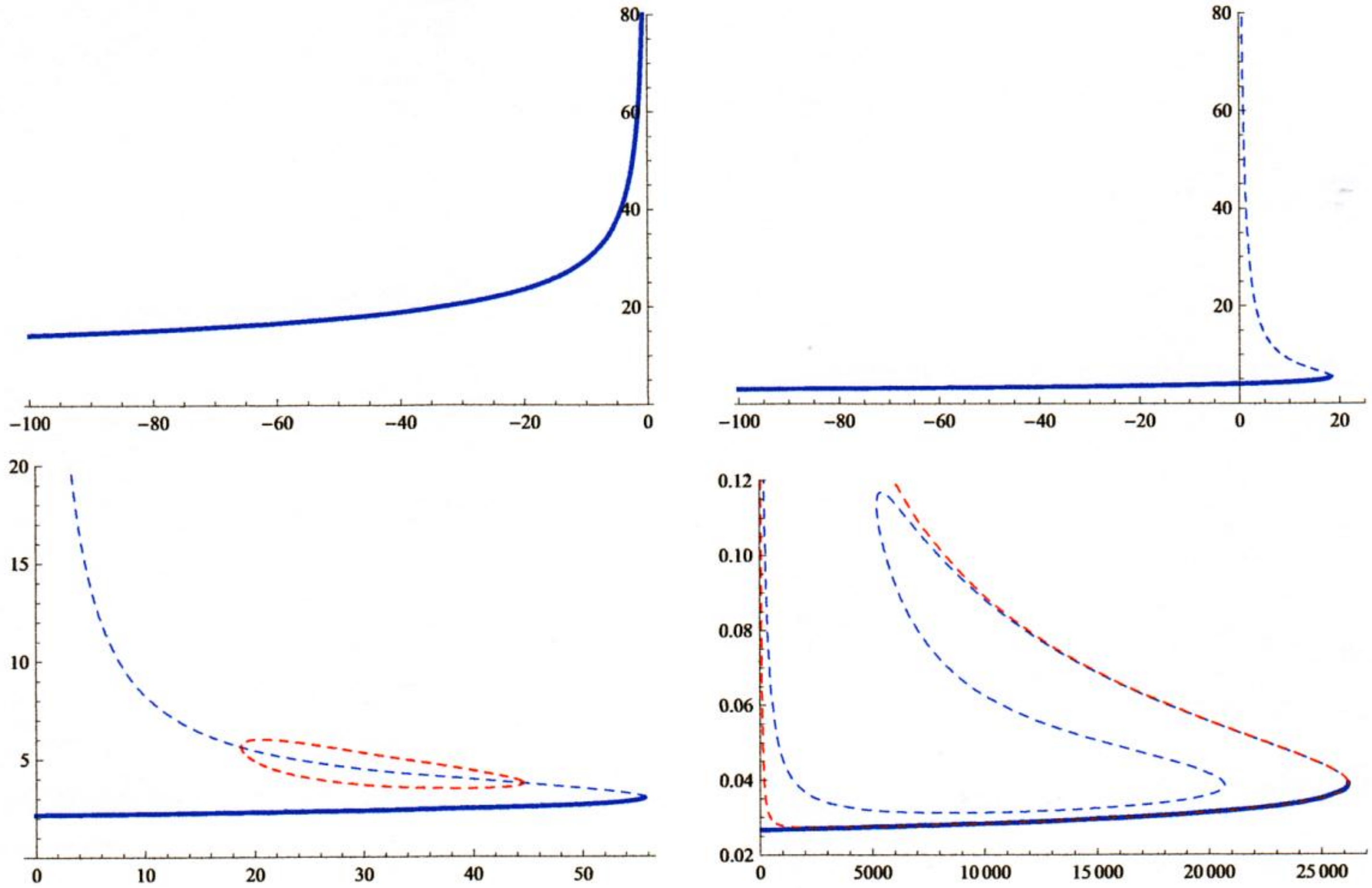


Figure 1.1: Bifurcation diagrams for $\lambda = 70$, $\lambda = -100$, $\lambda = -150$ and $\lambda = -750$, respectively.

Conclusions

- **The more polluted is the habitat, measured by the size of $|\lambda|$, the larger is the complexity of the dynamics.**

- **The more polluted is the habitat, the stronger are the squashing effects.**

- **In all circumstances, the unique stable steady-state is the minimal one.**