Complete Dynamics in a Heterogeneous Competition-Diffusion System

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Homogeneous Environment - Constant Coefficients

Lotka-Volterra competition-diffusion system in homogeneous environment:

$$\begin{cases} U_t = d_1 \Delta U + U(a_1 - b_1 U - c_1 V) & \text{in } \Omega \times (0, T) \\ V_t = d_2 \Delta V + V(a_2 - b_2 U - c_2 V) & \text{in } \Omega \times (0, T) \\ \partial_\nu U = \partial_\nu V = 0 & \text{on } \partial\Omega \times (0, T) \end{cases}$$

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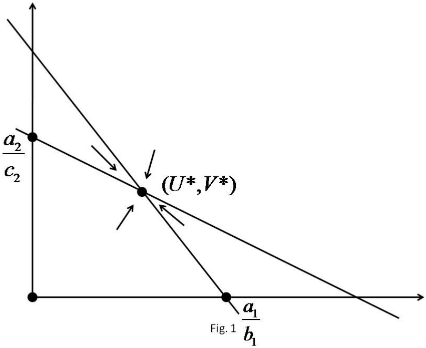
$(U^*, V^*) = (\frac{a_1 c_2 - a_2 c_1}{b_1 c_2 - b_2 c_1}, \frac{b_1 a_2 - b_2 a_1}{b_1 c_2 - b_2 c_1})$



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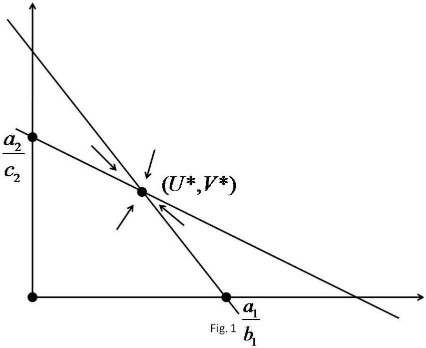

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Fact: (U^*, V^*) is *globally asymp stable* in $[U > 0, V > 0]$.
(In particular, U and V will always co-exist!)

Weak Competition in Heterogeneous Environment

Consider

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where $m_i(x) \geq 0, i = 1, 2$, **nonconstant**, and $U_0 \not\equiv 0$, $V_0 \not\equiv 0$.

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Likewise we'll focus on the "weak-competition" case, i.e.

$$\frac{b_1}{b_2} > \frac{m_1(x)}{m_2(x)} > \frac{c_1}{c_2} \text{ for all } x \in \overline{\Omega}.$$

Heterogeneous Environment

- *Both* d_1, d_2 *large* $\Rightarrow \exists!$ s.s. which is *globally asymp. stable*, and tends to $\left(\frac{\overline{m}_1 c_2 - \overline{m}_2 c_1}{b_1 c_2 - b_2 c_1}, \frac{b_1 \overline{m}_2 - \overline{m}_1 b_2}{b_1 c_2 - b_2 c_1}\right)$ as $d_1, d_2 \to \infty$, where $\overline{m}_i = \frac{1}{|\Omega|}\int_\Omega m_i(x)dx$. [Lou, private communication]

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- So far, the situation seems similar to that of the constant coefficients case.
- *However, the remaining case, namely, when* d_1, d_2 *are not very small nor very large, is drastically different from its counter part in the constant coefficients case.*

Single Equation in Heterogeneous Environment

(I) In a heterogeneous environment $m(x) \geq 0$, **nonconstant**

$$\begin{cases} u_t = d\Delta u + u(m(x) - u) & \text{in } \Omega \times (0, T), \\ \partial_\nu u = 0 & \text{on } \partial\Omega \times (0, T). \end{cases}$$

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(ii) If $\int_\Omega h(x) < 0$, then the same conclusion in (i) holds for all $0 < d < 1/\lambda_1(h)$, where $\lambda_1(h)$ is the nonzero principal eigenvalue of

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Lemma

The problem has a nonzero principal eigenvalue $\lambda_1 = \lambda_1(h)$ iff h changes sign and $\int_\Omega h \neq 0$. More precisely, if h changes sign, then

1. $\int_\Omega h = 0 \Leftrightarrow$ *0 is the only principal eigenvalue.*
2. $\int_\Omega h > 0 \Leftrightarrow \lambda_1(h) < 0$.
3. $\int_\Omega h < 0 \Leftrightarrow \lambda_1(h) > 0$.
4. $\lambda_1(h_1) > \lambda_1(h_2)$ *if* $h_1 \leq h_2$, $h_1 \not\equiv h_2$, *and* h_1, h_2 *both change sign.*
5. $\lambda_1(h)$ *is continuous in h; i.e.* $\lambda_1(h_\ell) \to \lambda_1(h)$ *if* $h_\ell \to h$ *in* $L^\infty(\Omega)$.

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Moreover, $\int_\Omega \theta_{d,m} \to \int_\Omega m(x)$ as $d \to 0$ or ∞, since

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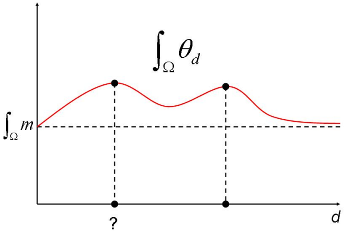

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Theorem (Lou; JDE (2006))

Suppose $m_1(x) = m_2(x) \geq 0$. Then $\forall b \in (b_*, 1)$, there exists $\overline{c} \in (0, 1]$ small such that if $c \in (0, \overline{c})$, $(\theta_{d_1}, 0)$ is globally asymp stable for some $d_1 < d_2$, where $b_* = \inf_{d>0} \int_\Omega m / \int_\Omega \theta_d$.

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In particular, for some $0 < b, c < 1$ and d_1, d_2, U will wipe out V, and co-existence is no longer possible even when the competition is weak! *A remarkable theorem!*

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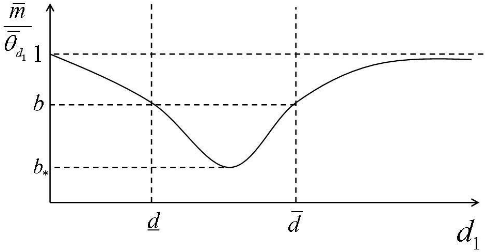

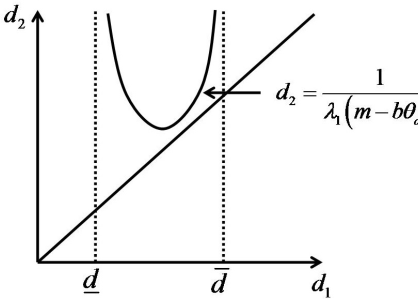

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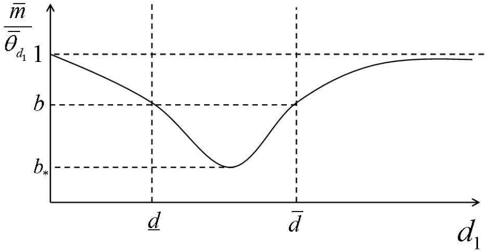

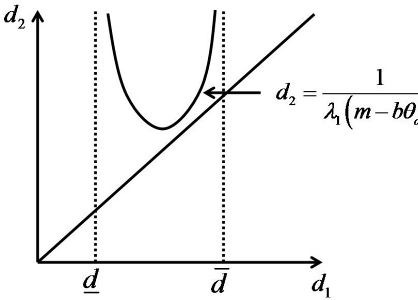

- $b > b_*$, c small, for above $d_1, d_2 \Rightarrow$ no co-existence

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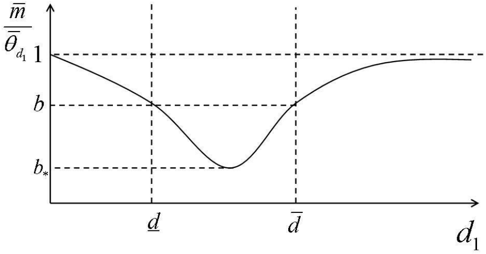
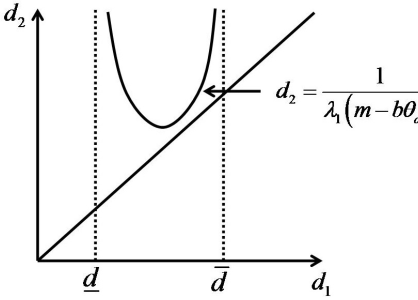
- $b < b_* \Rightarrow (\theta_{d_1}, 0)$ unstable (regardless of d_1, d_2, c)
- $b > b_* \Rightarrow (\theta_{d_1}, 0)$ stable for $d_1 \in (\underline{d}, \overline{d})$ and $d_2 > 1/\lambda_1(m - b\theta_{d_1})$





- $b > b_*$, c small, for above $d_1, d_2 \Rightarrow$ no co-existence
- $(0, \theta_{d_2})$ unstable if $d_1 < d_2$ (indep of b, c)

Complete Dynamics: $d_1 \leq d_2, 0 < b < 1, c$ small

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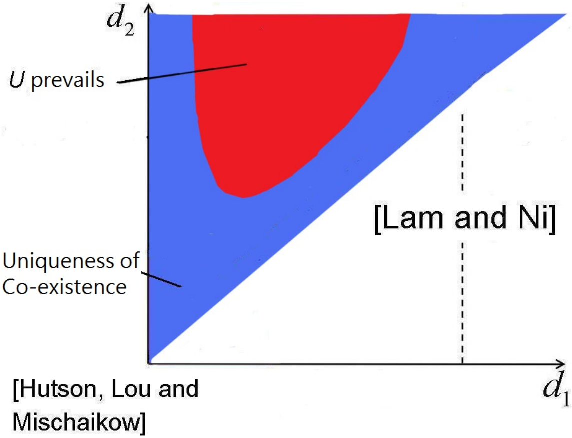
Note: c^* is uniform in (indep of) $b \in (0, 1)$.

Complete Dynamics

- $d_1 \le d_2, b > b_*, c$ small, $\overline{\Sigma}_b$ (the region, including its boundary, where U always wipes out V regardless of initial values) is colored in red



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- Now $\lambda_k \leq 0 \Rightarrow$ *either* $|\Phi_k| < |\Psi_k|$ *pointwise or* $|\Phi_k|_{L^2} \leq \epsilon |\Psi_k|_{L^2}$ *depending on the various cases above.*

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- For instance, consider the case (A) when both $d_1, d_2 \to 0$:

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To get a contradiction, suffices to show

$$\int_\Omega (b \frac{V}{\|V\|_{L^\infty(\Omega)}} \Phi\Psi + \frac{V}{\|V\|_{L^\infty(\Omega)}} \Psi^2) > 0,$$

which follows from $\Phi + \Psi < 0$.

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To see that $\Phi + \Psi < 0$, suffices to show (by max principle)

$$\begin{cases} d_1 \Delta(\Phi + \Psi) + (m - 2U - cV + \lambda)(\Phi + \Psi) > 0 & \text{in } \Omega, \\ \partial_\nu(\Phi + \Psi) \leq 0 & \text{on } \partial\Omega, \end{cases}$$

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which is equivalent to

$$\begin{aligned} & d_1 \Delta(-\Psi) + (m - 2U - cV + \lambda)(-\Psi) - cU\Psi \\ & = \frac{d_1}{d_2}\left[-bV\Phi + (m - bU - 2V)\Psi + \lambda\Psi\right] - (m - 2U - cV + \lambda)\Psi - cU\Psi \\ & = -\frac{d_1}{d_2} bV\Phi - \lambda\left(1 - \frac{d_1}{d_2}\right)\Psi \\ & + \left[\frac{d_1}{d_2}(m - bU - 2V) - (m - 2U - cV) - cU\right]\Psi < \ 0 \end{aligned}$$

since $U \to m$, $V \to (1 - b_\infty)m$, $0 < d_1 \le d_2$ and $c \searrow 0$, the terms in the square bracket converge to $\left[1 - \frac{d_1}{d_2}(1 - b_\infty)\right] m > 0$ in $\bar{\Omega}$, while the first two terms are non-positive.

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Another important ingredient in our proof: Given $d > 0$, $h \in L^\infty(\Omega)$ may change sign, let $\mu_1(d,h)$ be 1st eigenvalue of

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Proposition

Let $\mu_1(d_k,h_k) < 0$ for all k (i.e. $\theta_k := \theta_{d_k,h_k} > 0$ exists) where $h_k \in C(\bar{\Omega})$ and $d_k > 0$ and $\lim_{k\to\infty} h_k = h_\infty$ in $C(\bar{\Omega})$. Then

(a) *if $d_k \to 0$, then $\theta_k \to \max\{h_\infty, 0\}$ in $L^\infty(\Omega)$;*

(b) *if $d_k \to \infty$, then $\theta_k \to \overline{h}_\infty$ and $\tilde{\theta}_k := \theta_k/\|\theta_k\|_{L^\infty(\Omega)} \to 1$ in $L^\infty(\Omega)$;*

(c) *if $d_k \to d_\infty \in \mathbb{R}^+$, then $\mu_1(d_\infty, h_\infty) \leq 0$. Moreover,*
   - (i) *if $\mu_1(d_\infty,h_\infty) = 0$, then $\theta_k \to 0$ and $\theta_k/\|\theta_k\|_{L^\infty(\Omega)} \to \psi_1$ in $L^\infty(\Omega)$, where ψ_1 is the 1st eigenfcn (normalized) corresp to $\mu_1(d_\infty,h_\infty)$.*
   - (ii) *if $\mu_1(d_\infty,h_\infty) < 0$, then $\theta_k \to \theta_{d_\infty,h_\infty}$.*

Progress: For $0 \le c \le 1$ [Lam and Ni: SIAP (2012)]

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Remark: Interesting that c could be *even* bigger than b in (I).

Progress: For $0 \le c \le 1$ [Lam and Ni: SIAP (2012)]

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$$\begin{cases} d_1 \Delta U + U(m(x) - U - cV) = 0 & \text{in } \Omega \\ d_2 \Delta V + V(m(x) - bU - V) = 0 & \text{in } \Omega \\ \partial_\nu U = \partial_\nu V = 0 & \text{on } \partial\Omega \end{cases}$$

(I) *For any* ϵ, $\exists \delta(\epsilon) > 0$ *s.t. for* $1 - \delta < b < 1$, $0 \le c \le 1$, $\epsilon < d_1 < 1/\epsilon$ *and* $d_2 \ge d_1 + \epsilon$, $(\theta_{d_1}, 0)$ *is globally asymp. stable.*

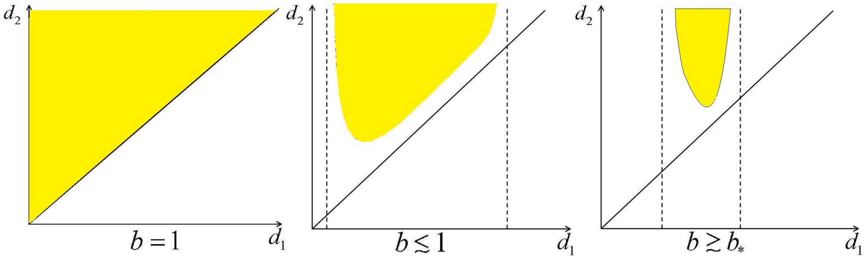
Remark: Interesting that c could be *even* bigger than b in (I).

(II) *For* $0 < b, c < 1$, $\exists \epsilon > 0$ s.t. *if* $|d_1 - d_2| < \epsilon$ *then* $\exists$ *unique positive s.s.* $(\tilde{U}, \tilde{V})$. *Moreover,* $(\tilde{U}, \tilde{V})$ *is globally asymp. stable; and if* $d_1, d_2 \to d > 0$, *then*

$$(\tilde{U}, \tilde{V}) \to \frac{1}{1 - bc} \begin{pmatrix} 1 - c \\ 1 - b \end{pmatrix} \theta_d.$$

Local stability of $(\theta_{d_1}, 0)$: $b > b_*, 0 \le c \le 1$

Now we vary b as a parameter. Then for any $c \in [0, 1]$, Lou obtained the following picture: (the regions shaded yellow represent the (d_1, d_2) for which $(\theta_{d_1}, 0)$ is locally asymp. stable.)



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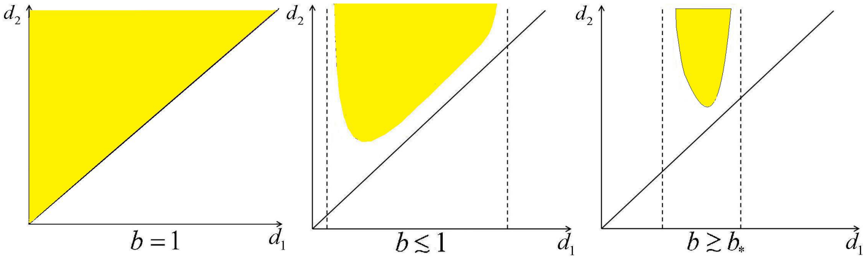

- As $b \nearrow 1$, $\Sigma_b \nearrow \mathcal{Q} \equiv \{(d_1, d_2) \in \mathbb{R}^+ \times \mathbb{R}^+ : d_1 < d_2\}$

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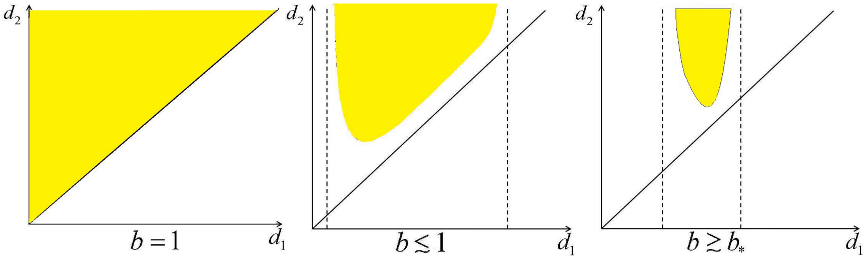
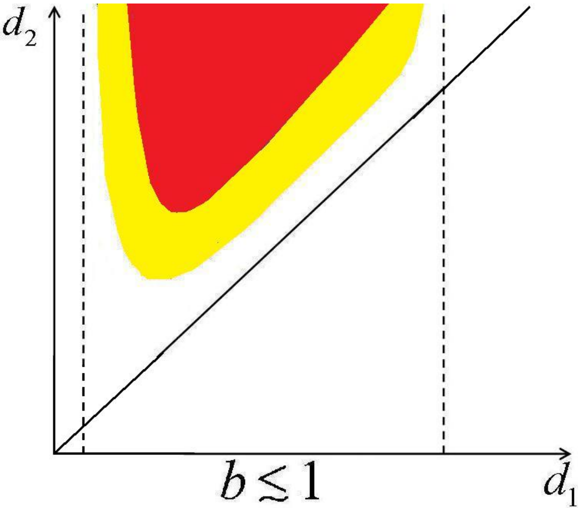


- As $b \nearrow 1, \Sigma_b \nearrow \mathcal{Q} \equiv \{(d_1, d_2) \in \mathbb{R}^+ \times \mathbb{R}^+ : d_1 < d_2\}$
- For $(d_1, d_2) \in \mathcal{Q} \setminus \overline{\Sigma_b}$ (i.e. *white area in the upper triangular region*), there is at least one locally stable co-existence s. s.

Globally Stable Coexistence S.S.: $b > b_*, 0 \leq c \leq 1$

The regions shaded blue represent the (d_1, d_2) for which there exists a unique coexistence s.s. which is globally asymp. stable.



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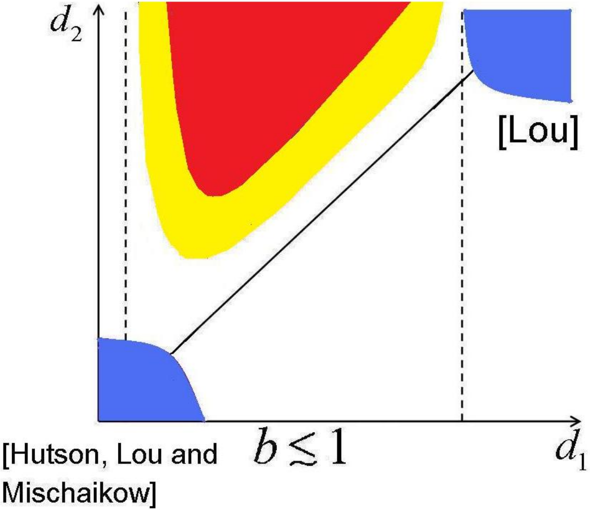

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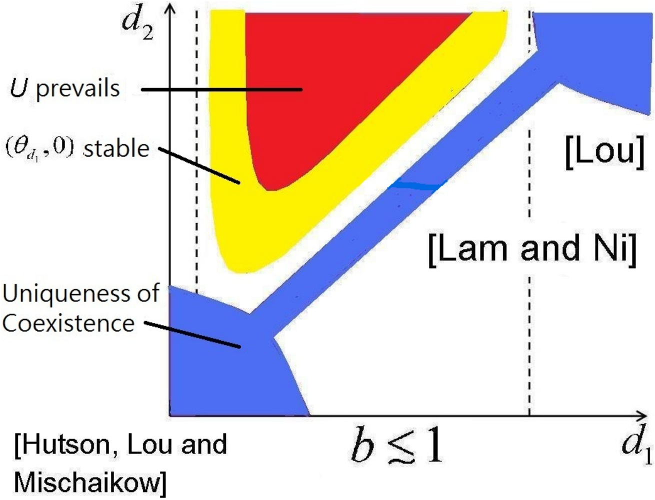

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(I) *For* $0 < d_1 < d_2$, $\exists\delta > 0$ *s.t. for* $1 - \delta < b < 1$ *and* $0 \le c \le 1$, $(\theta_{d_1}, 0)$ *is globally asymp. stable.*

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- However, the three semitrivial/trivial solutions are stable with respect to perturbation: the principal eigenvalue of the linearized system at those solutions are non-zero.
- (Given each semitrivial/trivial solution, there exists a neighborhood independent of b_k so that the semitrivial/trivial solution is the only solution in the neighborhood for all k large.)
- Therefore, we must have $(U_k, V_k) \in S$ for all k large. Contradicting the assumption that (U_k, V_k) is a coexistence s.s.

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(II) *For* $0 < b, c < 1$, $\exists \epsilon > 0$ s.t. *if* $|d_1 - d_2| < \epsilon$ *then* $\exists$ *unique positive s.s.* $(\tilde{U}, \tilde{V})$. *Moreover,* $(\tilde{U}, \tilde{V})$ *is globally asymp. stable; and if* $d_1, d_2 \to d > 0$, *then*

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Now perturb d_1 and d_2 in the same way as in the proof of (I).

Slower diffuser always prevails?

Consider a special case

$$
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V_t = d_2 \Delta V + V(m(x) - bU - V) & \text{in } \Omega \times (0, T) \\
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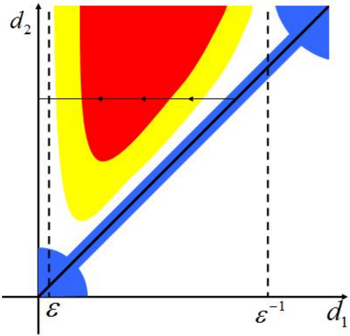

Conjecture

Finally,

Conjecture [Lou; JDE (2006)]: $(\theta_{d_1}, 0)$ *is globally asymp stable for* $b > b_*$, $c \in (0, 1)$ *and* $(d_1, d_2) \in \Sigma_b$.